

Small Oscillations

The Eigenvalue Equation and Normal Coordinates

Consider a system undergoing free, small oscillations in the s -dimensional space. Let $q_1, q_2, \dots, q_s$ be its s generalized coordinates. We consider conservative systems only so that the potential energy U is a function of the coordinates q_i 's only. Let U have a minimum for $q_i = q_{i0}$,

$$i = 1, 2, \dots, s,$$

$$\text{Put } x_i = q_i - q_{i0}; \quad i = 1, 2, \dots, s. \quad \text{--- (1)}$$

Now, the potential is expanded as a Taylor series and we retain non-zero terms upto the second power in x_i and neglect all higher order terms in the small oscillations limit.

$$\text{We know that } U(x) = \frac{1}{2} k x^2 \quad \text{--- (2)}$$

Generalization of eq. (2) is

$$U(x_1, x_2, \dots, x_s) = \frac{1}{2} \sum_{i,k=1}^s K_{ik} x_i x_k \quad \text{--- (3)}$$

Where, for simplicity, we have chosen the minimum value of U as zero. [Since the factor $x_i x_k \rightarrow$ itself under the interchange $i \leftrightarrow k$ we may take $K_{ik} = K_{ki}$, i.e., we assume symmetrical coefficients in eq. (3).] The kinetic energy of the system in terms of generalized velocities $\dot{q}_i$'s can be written as

$$T = \frac{1}{2} \sum_{i,k=1}^s a_{ik}(q) \dot{q}_i \dot{q}_k$$

In the approximation of small oscillations we may put $a_{ik}(q) \approx a_{ik}(q_0) = m_{ik}$.

$$\frac{d^2x}{dt^2} = -\frac{Kx}{m}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Then, T can be written as

$$T = \frac{1}{2} \sum_{i,k=1}^s m_{ik} \dot{x}_i \dot{x}_k \quad \text{--- (4)}$$

Again, we take symmetrical coefficients, $m_{ik} = m_{ki}$. Thus the Lagrangian L of a system with many degrees of freedom and executing free, small oscillations can be written as

$$L = T - V, \quad L = \frac{1}{2} \sum_{i,k} [m_{ik} \dot{x}_i \dot{x}_k - k_{ik} x_i x_k] \quad \text{--- (5)}$$

In order to derive the equations of motion, we write down the total differential of the Lagrangian,

$$dL = \frac{1}{2} \sum_{i,k} [m_{ik} \dot{x}_i d\dot{x}_k + m_{ik} \dot{x}_k dx_i - k_{ik} dx_k - k_{ik} x_k dx_i]$$

Since the summation is over both i and k , we can interchange i and k in the first and third terms:

$$dL = \frac{1}{2} \sum_{i,k} [m_{ki} \dot{x}_k d\dot{x}_i + m_{ik} \dot{x}_k dx_i - k_{ki} x_k dx_i - k_{ik} x_k dx_i]$$

$$= \sum_{i,k} [m_{ik} \dot{x}_k d\dot{x}_i - k_{ik} x_k dx_i],$$

Since $m_{ki} = m_{ik}$, $k_{ki} = k_{ik}$.

Thus,

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_k m_{ik} \dot{x}_k, \quad \frac{\partial L}{\partial x_i} = - \sum_k k_{ik} x_k$$

and the Lagrange's equations give

$$\sum_{k=1}^s [m_{ik} \ddot{x}_k + k_{ik} x_k] = 0; \quad i = 1, 2, \dots, s \quad \text{--- (6)}$$

Equations (6) are the desired equations of motion, valid only in the small oscillations approximation. Further, these s equations are all linear, homogeneous differential equations with constant coefficients. We have to solve these simultaneously to obtain the motion near equilibrium.

Equations (6) are similar to those appearing in electrical circuit theory. We try for a solution in the form

$$x_k = A_k \exp(i\omega t), \quad k = 1, 2, \dots, s \quad (7)$$

In other words, we look for eigenvibrations of the system, that is such solutions where all x_k oscillate with the same frequency ω . Here, A_k are undetermined constants. Substituting in Eqs. (6) we obtain

$$\sum_k (-\omega^2 m_{ik} + k_{ik}) A_k = 0, \quad i = 1, 2, \dots, s.$$

There are s linear, homogeneous equations for the A_k and a non-trivial solution exists only if the determinant of the coefficients vanishes:

$$\begin{vmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} & \dots & \dots \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ k_{ss} - \omega^2 m_{ss} & \dots & \dots & \dots \end{vmatrix} = 0, \quad (8)$$

$$\text{or,} \quad |k_{ik} - \omega^2 m_{ik}| = 0.$$

This condition on the determinant results in the characteristic equation which is an algebraic equation of the s^{th} degree in ω^2 . The roots of the characteristic equation provide the frequencies for which Eq. (7) give a correct solution to the equation of motion. In general, Eq. (8) has s different, real, positive roots ω_k^2 , $k=1, 2, \dots, s$. However, in some cases all of these roots may not be different. ω_k^2 's are called the characteristic frequencies or eigenfrequencies of the given system.

Now, we shall prove that the characteristic equation has real and positive roots. This can be seen either from pure physical arguments or in the following mathematical manner: We have

$$\sum_k (-\omega^2 m_{ik} + k_{ik}) A_k = 0,$$

$$\sum_i \sum_k A_i^* (-\omega^2 m_{ik} + k_{ik}) A_k = 0,$$

$$\omega^2 = \frac{\sum_i \sum_k k_{ik} A_i^* A_k}{\sum_i \sum_k m_{ik} A_i^* A_k}$$

Since both k_{ik} 's and m_{ik} 's are real and symmetric; it follows that the quadratic forms in both the numerator and the denominator of this expression are real. $[(\sum_i \sum_k k_{ik} A_i^* A_k)]^* = \sum_i \sum_k k_{ik}^* A_i A_k^* = \sum_i \sum_k k_{ik} A_i A_k^* = \sum_i \sum_k k_{ki} A_i^* A_k$
 $A_i A_k^* = \sum_i \sum_k k_{ki} A_k A_i^* = \sum_i \sum_k k_{ik}^* A_i^* A_k$

Thus, ω^2 is real. Further, both the numerator and the denominator are also positive definite.