Numerical Integration:-

Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly (clearly), it is required to compute the value of the definite integral

$$I = \int_a^b f(x)\,dx \quad \text{---} \quad (1)$$

Definite integral is related to summation (it is a limit of sums of a certain kind). As in the case of numerical differentiation, one replaces $f(x)$ by an interpolating polynomial $\phi(x)$ and finds value, for numerical integration, an approximate value of the definite integral.

Thus, different integration formulae can be obtained depending upon the type of the interpolating formula used.

We derive, a general formula for numerical integration using Newton's forward difference formula.

Let the interval $[a, b]$ be divided into n equal subintervals such that

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

clearly $x_n = x_0 + nh$

Hence the integral

$$I = \int_{x_0}^{x_n} y\,dx$$

Approximating y by Newton's Forward difference formula, we obtain.

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots \right] dx \quad \text{---} \quad (1)$$

Since $x = x_0 + ph$

$dx = h\,dp$

Then Eph (1)

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots \right] dp$$

which gives on simplification

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad (2)$$

From this general formula, we can obtain different integration formulae by putting $n = 1, 2, 3,$ etc.

Trapezoidal Rule:-

Putting $n = 1$ in general formula (2), all higher differences will become zero than the first difference. then

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}(y_0 + y_1) \quad \text{---} \quad (3)$$

For the next interval $[x_1, x_2]$, we obtain similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_1 + y_2] \quad \text{---} \quad 4$$

And so, on the last interval $[x_{n-1}, x_n]$

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \quad \text{---} \quad (5)$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots y_{n-1}) + y_n \right]$$

which is known as the Trapezoidal Rule

Exp. Find the area bounded by the curve and the x-axis from $x = 7.47$ to $x = 7.52$, from the following table.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$ y_i	1.93	1.95	1.98	2.01	2.03	2.06

Solution we know that

$$\text{Area} = \int_{x_0}^{x_n} f(x)\,dx$$

with $h = 0.01$, $x_0 = 7.47$ and $x_n = 7.52$

$$\text{Area} = \int_{7.47}^{7.52} y\,dx$$

The Trapezoidal rule gives

$$\text{Area} = \frac{0.01}{2}\left[1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06\right]$$

$$= \frac{0.01}{2}\left[1.93 + 2 \times 7.97 + 2.06\right]$$

$$= \frac{0.01}{2}\left[1.93 + 15.94 + 2.06\right]$$

$$= \frac{0.01}{2} \times 19.93$$

$$= 0.09965$$

Area. $= .0996$