Numerical Differentiation:

Numerical Differentiation formula is deriving to differentiate the interpolating polynomial. Here, we illustrate the derivation with Newton's forward difference formula.

So, consider Newton's forward difference formula:-

$$y = y_0 + \frac{p}{\lfloor 1}\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots \quad (1)$$

where $x = x_0 + ph$ — (2) $\Rightarrow \frac{dp}{dx} = \frac{1}{h}$

Then $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right]$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right] \quad (3)$$

This formula can be used for calculating the value of dy/dx for non-tabular values of x. For tabular values of x, by setting $x = x_0$ then $x = x_0 + ph$ gives $p = 0$ and hence.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots + \frac{(-1)^{n-1}}{n}\Delta^n y_0\right] \quad (4)$$

Again differentiating Eqn (3) then gives

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + \cdots\right] \quad (5)$$

Thus, $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\right]$ (6)

Therefore, formula for computing higher derivatives may be obtained by successive differentiation.

In similar way, Newton's backward difference formula:

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{\lfloor 2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{\lfloor 3}\nabla^3 y_n + \cdots \quad \text{---(i)}$$

gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right] \quad \text{---(ii)}$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right] \quad \text{---(iii)}$$

Similarly, Stirling's (Central difference) formula —

$$y = y_0 + p\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{\lfloor 2}\Delta^2 y_{-1} + \frac{p(p^2-1)}{\lfloor 3}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{p^2(p^2-1)}{\lfloor 4}\Delta^4 y_{-2} + \cdots \quad \text{---(1)}$$

gives

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \cdots\right] \quad \text{---(2)}$$

and

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \cdots\right] \quad \text{---(3)}$$

If a derivative is required near the end of a table, one of the following formula may be used to obtain better accuracy.

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$$hy'_0 = \left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \cdots + (-1)^{n-1}\frac{\Delta^n}{n}\right]y_0 \quad \text{---(A1)}$$

Also,

From Str. $hy'_0 = \left(\Delta + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \frac{1}{12}\Delta^4 - \frac{1}{20}\Delta^5 + \frac{1}{30}\Delta^6 - \cdots\right) y_{-1}$ — (A2)

HFD $hy'_0 = \left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \cdots\right) y_0$ — (A3)

STR. $= \left(\Delta^2 - \frac{1}{12}\Delta^4 + \frac{1}{12}\Delta^5 - \frac{13}{180}\Delta^6 + \frac{11}{180}\Delta^7 - \frac{29}{560}\Delta^8 + \cdots\right) y_{-1}$ — (A4)

HBD $hy'_n = \left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right) y_n$ — (A5)

$= \left(\nabla - \frac{1}{2}\nabla^2 - \frac{1}{6}\nabla^3 - \frac{1}{12}\nabla^4 - \frac{1}{20}\nabla^5 - \frac{1}{30}\nabla^6 - \cdots\right) y_{n+1}$ — (A6)

$h^2 y''_n = \left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + \cdots\right) y_n$ (A7)

$= \left(\nabla^2 - \frac{1}{12}\nabla^4 - \frac{1}{12}\nabla^5 - \frac{13}{180}\nabla^6 - \frac{11}{180}\nabla^7 - \frac{29}{560}\nabla^8 - \cdots\right) y_{n+1}$ — (A8)

Exp: From the following table of values of x and y obtain dy/dx and d^2y/dx^2 for $x = 1.2$

Sol. The difference table is

	x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
	1.0 (x_{-1})	2.7183 (y_{-1})	0.6018 (Δy_{-1})					
At $x_0 = 1.2$	1.2 (x_0)	3.3201 (y_0)	0.7351 (Δy_0)	0.1333 ($\Delta^2 y_{-1}$)	0.0294 ($\Delta^3 y_{-1}$)			
	1.4	4.0552	0.8978	0.1627 ($\Delta^2 y_0$)	0.0361 ($\Delta^3 y_0$)	0.0067 ($\Delta^4 y_{-1}$)	0.0013 ($\Delta^5 y_{-1}$)	
At $x = 1.6$		4.9530		0.1988	0.0441	0.0080 ($\Delta^4 y_0$)	0.0014 ($\Delta^5 y_0$)	0.0001
			1.0966					
	1.8	6.0496		0.2429 ($\nabla^2 y_n$)	0.0535 ($\nabla^3 y_n$)	0.0094 ($\nabla^4 y_n$)	($\nabla^5 y_n$)	
			1.3395 (∇y_n)					
At $x_n = 2.0$		7.3891 (y_n)	(∇y_{n-1})	0.2964 ($\nabla^2 y_n$)				
			1.6359 (∇y_n)					
At $x_n = 2.2$		9.0250 (∇y_n)						

Here $x = x_0 = 1.2 \Rightarrow y_0 = 3.3201$ and $h = 0.2$

Then using formula for first derivative of the function.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots \frac{(-1)^{n-1}}{n}\Delta^n y_0\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0\right.$$

and $\Delta y_0, \Delta^2 y_0, \cdots$ values from difference table.

$$\left[\frac{dy}{dx}\right]_{x=1.2} = 3.3205$$

If we use formula of stirling's (A2) then

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_{-1} + \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-1} + \frac{1}{12}\Delta^4 y_{-1} - \frac{1}{20}\Delta^5 y_{-1}\right.$$

Then uses difference table values as

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.6018 + \frac{1}{2}(0.1333) - \frac{1}{6}(0.0274) + \frac{1}{12}(0.0067) - \frac{1}{20}(0.0013)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = 3.3205$$ As same above value.

Similarly from (A3) we obtain

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = 3.318$$

Again using formula (A4) then.

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1333 - \frac{1}{12}(0.0067) + \frac{1}{12}(0.0013)\right.$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = 3.32$$