Exp. Calculate the first and seconds derivatives of the function tabulated in the previous Example at the point $x = 2.2$ and also dy/dx at $x = 2.0$

Solution - For this, we use the table of differences of previous example.

Here $x_n = 2.2$, $y_n = 9.0250$, and $h = 0.2$

Use formula A5 for this

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014) + \frac{1}{6}(0.0001)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + 0.1482 + 0.0178 + 0.00235 + 0.00028 + 0.000017\right]$$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = 9.0228$$

For $\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n\right]$

Then $\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014) + \frac{137}{180}(0.0001)\right]$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + 0.00862 + 0.00112 + 0.000076\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = 8.992$$

To find $\frac{dy}{dx}$ at $x = 2.0$, we can use this formula

(A5) $$\left[\frac{dy}{dx}\right]_{x=2.0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n\right]$$

or (A6) $$\left[\frac{dy}{dx}\right]_{x=2.0} = \frac{1}{h}\left[\nabla y_{n+1} - \frac{1}{2}\nabla^2 y_{n+1} - \frac{1}{6}\nabla^3 y_{n+1} - \frac{1}{12}\nabla^4 y_{n+1} - \frac{1}{20}\nabla^5 y_{n+1}\right]$$

From A6 formula

$$\left[\frac{dy}{dx}\right]_{x=2.0} = \frac{1}{0.2}\left[1.3395 + \frac{1}{2}(0.2429) - \frac{1}{6}(0.0441) + \frac{1}{5}(0.0080) + \frac{1}{6}(0.0013) + \frac{1}{6}(0.0005)\right]$$

$$= 7.3896$$

and use A6 formula then

$$\left[\frac{dy}{dx}\right]_{x=2.0} = \frac{1}{0.2}\left[1.6359 - \frac{1}{2}(0.2764) - \frac{1}{6}(0.0536) - \frac{1}{12}(0.0094) - \frac{1}{20}(0.0044)\right]$$

$$= 7.3896$$

Exp. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$ from the same tabulated function of preceding Exampl.

Sol. Let $x_0 = 1.6$ it is middle point of table differences values then use Stirling formula.

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\frac{0.8978 + 1.0966}{2} - \frac{1}{12}(0.0361 + 0.0441) + \frac{1}{60}(0.0013 + 0.0011)\right]$$

$$= \frac{1}{0.2}\left[0.9972 - 0.0067 + 0.0005\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.6} = 4.9527 \text{ or } 4.9530$$

Exp. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x=1.6$ for the tabulated function of the difference table. Estimate the errors in these values.

Sol. – Choosing $x_0 = 1.6$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
$x_0 \Rightarrow$ 1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Truncation error $T_1 = \left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{6h}\left[\Delta^3 y_1 + \frac{\Delta^3 y_0}{2}\right]$

and $T_2 = \left[\frac{d^2x}{dx^2}\right]_{x=x_0} = \frac{1}{12h^2}\left[\Delta^4 y_{-1}\right]$

Value of $\frac{dy}{dx}$ at $x = 1.6$

Truncation error $T_1 = \left[\frac{d^4y}{dx}\right]_{x=1.6} = \frac{1}{6(0.2)}\left[\frac{0.03691 + 0.01667}{2}\right]$

$T_1 = 0.00342$

Rounding error $= \frac{3\epsilon}{2h} = \frac{3(0.5 \times 10^{-4})}{0.4}$

$= 0.00038$

Total error $= 0.00342 + 0.00038$

$\ln\left[\frac{dy}{dx}\right]_{x=1.6} = 0.0038$

$T_2 = \frac{1}{12h^2}\left[\Delta^4 y_{-2}\right] = \frac{1}{12 \times (0.04)}(0.00084)$

$= 0.01667$

Rounding error $= \frac{4\epsilon}{h^2} = \frac{4 \times 0.5 \times 10^{-4}}{0.04}$

$= 0.00050$

Total error in $\left[\frac{d^2y}{dx^2}\right]_{x=1.6} = 0.01667 + 0.00050$

$= 0.0217$

Using Stirling's formula, with the first differences we obtain $\left(\frac{dy}{dx}\right)_{x=1.6} = \frac{\Delta y_{-1} + \Delta y_0}{2h} = \frac{0.8978 + 1.0}{2 \times 0.2}$

$\left(\frac{dy}{dx}\right)_{x=1.6} = 4.9860$

The exact value is 4.9530 from previous. So that the error in the above solution is

$\Rightarrow 4.9860 - 4.9530 = 0.0330$ which agrees with Total error.

Similarly, the value of $\frac{d^2y}{dx^2}$ at $x = 1.6$ using formula

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.6} = \frac{\Delta^2 y_{-1}}{h^2} = \frac{0.1988}{0.04} = 4.9700$$

So that error

$$= 4.9700 - 4.9525$$

$$= 0.0175$$