Non-linear Equations :-

A first order differential non-linear equation of the form :-

Eqn. (1) $\longrightarrow$ $F(x, y, p) = 0$, where $p = y'(x) = \frac{dy}{dx}$ and F is a function of three variables (x, y and p).

The function F possesses the properties :-

(i) $F(x, y, p)$ is continuous in all the arguments;

(ii) The derivatives $F_p'(x, y, p)$ exists and is non-zero;

(iii) there exists the derivative $|F_p'(x, y, p)| < M$.

→ Any solution of Eqn. (1) of the form $y = \varphi(x, c)$ in which c is an arbitrary constant is called a general solution of the equation.

→ Any solution that may be obtained from the general solution of Eqn. (1) by assigning particular value to the constant c is called a particular solution of that equation.

→ A solution of Eqn. (1) that cannot be obtained by assigning specific value to the arbitrary constant in the general solution is called a singular solution of that equation.

Thus, a singular solution may exist if even one of conditions of the uniqueness theorem (or properties) is violated. There is considered -d case $F_p'(x, y, p) = 0$ only.

Thus, there is a system of equations:-

$$F(x, y, p) = 0 \quad \text{---------} \quad (1)$$

$$F'_p(x, y, p) = 0 \quad \text{---------} \quad (2)$$

which determines so called singular set. Eliminating p from Eph(2) yields

$$y = \varphi_i(x), \quad i = 1, 2, 3, \ldots l$$

where l is number of solutions of Eph(2), φ_i are functions of x alone. Substituting each of the equations in Eph(1) we find some of its solutions.

$$y = \varphi_j(x), \quad j = 1, 2, \ldots k \leq l$$

where k is a number of solutions of Eq^n(1), φ_j are some functions from the functions φ_i.

A solution of Eph(1) of the form $y = \varphi(x)$, where φ is a function of x alone, is a singular solution of that equation if it is tangent in every point to a general solution of that equation. This singular solution is the envelope of the family of general solutions of Eph(1). Hence we substitute each of the solutions φ_j in the conditions of tangency of two curves

Singular Sol. $\leftarrow \varphi_j(x) = g(x, C) \rightarrow$ general Sol.

$$\varphi'_j(x) = g'(x, C)$$

Exp. Find the singular solution of the differential equation.

$$xy'^2 - 2y'y + 4x = 0$$

Sol. Given differential equation.

$$xy'^2 - 2y'y + 4x = 0 \quad \text{---} \quad (i)$$

Substituting $y' = p \Rightarrow dy = p\,dx$

From (i)

$$xp^2 - 2py + 4x = 0 \quad \text{---} \quad (ii)$$

$$y = \frac{x}{2}p + \frac{2x}{p}$$

Differentiating w.r.to x, we get

$$\frac{dy}{dx} = \frac{p}{2} + \frac{x}{2}\cdot\frac{dp}{dx} + \frac{2}{p} - \frac{2x}{p^2}\frac{dp}{dx}$$

$$p = \frac{p}{2} + \frac{2}{p} + \left(\frac{x}{2} - \frac{2x}{p^2}\right)\frac{dp}{dx}$$

$$\frac{p}{2} = \frac{2}{p} + \left(\frac{x}{2} - \frac{2x}{p^2}\right)\frac{dp}{dx}$$

$$p = \frac{4}{p} + \frac{1}{p^2}\left(\frac{2x}{2}p^2 - 4x\right)\frac{dp}{dx}$$

$$p^3 = 4p + (xp^2 - 4x)\frac{dp}{dx}$$

$$p^3 - 4p - (xp^2 - 4x)\frac{dp}{dx} = 0$$

$$p^3 - 4p - xp^2\frac{dp}{dx} + 4x\frac{dp}{dx} = 0$$

$$p^2\left(p - x\frac{dp}{dx}\right) - 4\left(p - x\frac{dp}{dx}\right) = 0$$

$$\left(p - x\frac{dp}{dx}\right)(p^2 - 4) = 0$$

$\therefore$ $p - x\frac{dp}{dx} = 0$ or $p^2 - 4 = 0$

$p = \pm 2$

$\therefore \frac{dp}{p} = \frac{dx}{x}$

Integrating both sides

$\int \frac{dp}{p} = \int \frac{dx}{x} + c$

$\log p = \log x + \log c$

$p = cx$

Putting $p = cx$ in Eqn (ii), then

$xp^2 - 2py + 4x = 0$

$xc^2x^2 + 2cxy + 4x = 0$

$c^2x^2 - 2cy + 4 = 0$

$y = (c^2x^2 + 4)/2c$ is its general solution ——— (3)

or putting $p = 2$ then

$4x - 2x^2y + 4x = 0 \Rightarrow y = 2x$

$p = -2, \quad 4x + 2x^2y + 4x \quad y = -2x$

According to system equations

$F(x, y, p) = xp^2 - 2py + 4x = 0$

$F_p'(x, y, p) = 2px - 2y = 0$

Eliminating p then yields

$y = 2x \equiv \varphi_1(x)$

$y = -2x \equiv \varphi_2(x)$

Substituting $\varphi_1(x)$ and $\varphi_2(x)$ in given equation we get its solutions. Hence $k=l=2$ then

$\pm 2x = \frac{c}{2}x^2 + \frac{2}{c}$ From Eq^2 (iii)

$\pm 2x = \frac{1}{2c}(c^2x^2+4)$

$(cx)^2 + 4 - 4xc = 0$

$(cx-2)^2 = 0$

$cx = 2$

$x = \pm 2/c$ are the singular solutions.