* Linear differential Equation:-

A differential equation is said to be Linear if the dependent variable (y) and its differential coefficients ($\frac{dy}{dx} = y'$) occur only in the first degree and not multiplied together (i.e $y\frac{dy}{dx}$ term not appeared in differential equation).

Thus, the standard form of a linear differential equation of the first order, commonly known as Leibnitz's linear equation-

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{---(1)}$$

Where P, Q are the functions of only x.

* If a function $Q(x) = 0$, then equation (1) takes the form $\frac{dy}{dx} + P(x)y = 0$ and is called homogeneous linear differential equation.

To Solve the Eqn (1), multiply by $e^{\int P dx}$ in both sides, so that we get-

$$\frac{dy}{dx} e^{\int P dx} + y(e^{\int P dx} P(x)) = Q(x) e^{\int P dx}$$

$$\frac{d}{dx}(y e^{\int P dx}) = Q(x) e^{\int P dx}$$

Integrating both sides.

$$\int \frac{d}{dx}(y e^{\int P dx}) = \int Q(x) e^{\int P dx} dx + C$$

Thus-

$$y e^{\int P dx} = \int Q(x) e^{\int P dx} dx + c \quad \text{---(iii)}$$

This is the required solution of the linear differential equation of first order.

The factor $e^{\int P dx}$ is called the integrating factor (I.F) of the equation

(i)

$$I.F = e^{\int P dx}$$

$$y.(I.F) = \int Q(x)(I.F)dx + c$$

$\Rightarrow$ If this is homogeneous linear differential equation then $Q(x)=0$

Then equation (ii) becomes

$$y e^{\int P dx} = 0 + c$$

$$y = c e^{-\int P dx}$$ where c is constant

This is solution of homogeneous linear differential equation.

* An equation is linear if the dependent variable y (or variables) and their derivatives appear linearly (i.e no higher powers/degrees), that are only as first powers and not multiply together.

Ex/2 Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Sol. Given

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

which is in Leibhitz's Equation form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

then

$$P(x) = -\frac{1}{x+1}$$

$$Q(x) = e^{x}(x+1)$$

$$I = \int P\,dx = -\int \frac{dx}{x+1} = -\log(x+1) = \log(x+1)^{-1}$$

Thus the solution of Eqn (1) is

$$y \cdot (I.F) = \int e^{3x}(x+1)\cdot(I.F)\,dx + c$$

$$I.F = e^{\int P\,dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

Thus -

$$\frac{y}{x+1} = \int e^{3x}(x+1)\cdot\left(\frac{1}{x+1}\right)dx + c$$

$$\frac{y}{x+1} = \int e^{3x}\,dx + c$$

$$\frac{y}{x+1} = e^{3x}\cdot\frac{1}{3} + c$$

$$y = \left(\frac{1}{3}e^{3x} + c\right)(x+1)$$

This is the solution of Eq^{n} (1).

Exp. Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Sol. Equation can be written as

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is in Leibnitz's equation form

$$P(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$x(I.F.) = \int Q(y)(I.F)\,dy$$

$$x\cdot(e^{\tan^{-1}y}) = \int \frac{\tan^{-1}y}{1+y^2}\cdot e^{\tan^{-1}y}\,dy + c$$

Put $\tan^{-1}y = t$, $\frac{1}{1+y^2}dy = dt$

$$= \int t\cdot e^{t}\,dt + c$$

$$x\,e^{\tan^{-1}y} = t\cdot e^{t} - e^{t} + c = (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}$$

This is solution