

Inverse Interpolation:-

Given a set of values of x and y , the process of finding the value of x for a certain value of y is called inverse interpolation. When the values of x are at unequal intervals, the most obvious way of performing this process is by interchanging x and y in Lagrange's or Aitken's method.

Lagrange's interpolation formula:-

Let $y(x)$ be continuous and differentiable $(n+1)$ times in the interval (a, b) . Given the $(n+1)$ points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ where the values of x need not necessarily be equally spaced, we want to find a polynomial of degree n , say $L_n(x)$ such that

$$L_n(x_i) = y(x_i) = y_i \quad (i=0, 1, 2, \dots, n)$$

The equation of a straight line (a linear polynomial) passing through two points (x_0, y_0) and (x_1, y_1) such as $L_1(x)$, we say

$$L_1(x) = \frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1$$

$$= l_0(x) y_0 + l_1(x) y_1$$

$$= \sum_{i=0}^1 l_i(x) y_i \quad \text{--- (1)}$$

$$\text{where } l_0(x) = \frac{x-x_1}{x_0-x_1} \text{ \& } l_1(x) = \frac{x-x_0}{x_1-x_0} \text{ --- (1b)}$$

$$\text{Thus } l_0(x_0)=1, l_0(x_1)=0, l_1(x_0)=0, l_1(x_1)=1$$

From (iii) gives relations and can be expressed in a more convenient form as

$$l_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{--- (iv)}$$

∴ Therefore $l_i(x)$ in Eqn (ii) also have property.

$$\sum_{i=0}^1 l_i(x) = l_0(x) + l_1(x) = \frac{x-x_1}{x_0-x_1} + \frac{x-x_0}{x_1-x_0} = 1$$

Equation (ii) is the Lagrange polynomial of degree one passing through two points (x_0, y_0) and (x_1, y_1) . Similarly, the Lagrange polynomial of degree two passing through three points (x_0, y_0) , (x_1, y_1) and (x_2, y_2) then

$$L_2(x) = \sum_{i=0}^2 l_i(x) y_i$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 \quad \text{--- (vi)}$$

Where $l_i(x)$ satisfy the conditions given in (iv) and (v).

To derive the general formula,

Let

$$L_n(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \quad \text{--- (vii)}$$

be the desired polynomial of the n th degree.

Substituting these conditions in (vii) we get

$$\left. \begin{aligned} y_0 &= a_0 + a_1x_0 + a_2x_0^2 + \dots + a_nx_0^n \\ y_1 &= a_0 + a_1x_1 + a_2x_1^2 + \dots + a_nx_1^n \\ &\vdots \\ y_n &= a_0 + a_1x_n + a_2x_n^2 + \dots + a_nx_n^n \end{aligned} \right\} \quad \text{--- (viii)}$$

Solution of the set of Eqn (viii)

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} \neq 0 \quad \text{--- (ix)}$$

is called the Vandermonde's determinant.

$$\text{as } (x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)(x_1 - x_2) \dots (x_1 - x_n) \dots (x_{n-1} - x_n).$$

Eliminating $a_0, a_1, a_2, \dots, a_n$ from Eqn (vii) & (viii)

$$\begin{vmatrix} L_n(x) & 1 & x & x^2 & \dots & x^n \\ y_0 & 1 & x_0 & x_0^2 & \dots & x_0^n \\ y_1 & 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_n & 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = 0 \quad \text{--- (x)}$$

Thus, $L_n(x)$ is a linear combination of $y_0, y_1, \dots, y_n$. Hence

$$L_n(x) = \sum_{i=0}^n l_i(x) y_i \quad \text{--- (XI)}$$

where $l_i(x)$ are polynomials in x of degree n . Since

$$L_n(x_j) = y_j \quad \text{for } j=0, 1, 2, \dots, n$$

then Eq (VI) gives

$$l_i(x_j) = 0 \quad \text{if } i \neq j$$

$$l_j(x_j) = 1 \quad \text{if } i = j$$

Thus

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} \quad \text{--- (XII)}$$

which obviously satisfies the conditions (i)

if we now set

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)$$

Then

$$\begin{aligned} \pi'_{n+1}(x) &= \frac{d}{dx} [\pi_{n+1}(x)]_{x=x_i} \\ &= (x_i-x_0)(x_i-x_1)\dots(x_i-x_n) \end{aligned} \quad \text{--- (XIII)}$$

$$l_i(x) = \frac{\pi_{n+1}(x)}{(x_i-x_0)\pi'_{n+1}(x)}$$

Thus

$$L_n(x) = \sum_{i=0}^n \frac{\pi_{n+1}(x)}{\pi'_{n+1}(x)} y_i \quad \text{--- (XIII)}$$

which is called Lagrange's interpolation formula. Interchanging x and y in Eq (XIII) then obtain inverse interpolation formula.

Exp. Certain corresponding values of x and $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ and $(307, 2.4871)$. Find $\log_{10} 301$.

Sol. $x = 301$

Lagrange's interpolation formula-

$$\log_{10} x = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Putting values of $x = 301$, $x_0 = 300$, $x_1 = 304$, $x_2 = 305$
 $x_3 = 307$, $y_0 = 2.4771$, $y_1 = 2.4829$, $y_2 = 2.4843$ and
 $y_3 = 2.4871$

$$\log_{10} 301 = \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)} (2.4771)$$

$$+ \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)} (2.4829)$$

$$+ \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)} (2.4843)$$

$$+ \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} (2.4871)$$

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} (2.4771) + \frac{(1)(-4)(-6)^2}{(4)(-1)(-3)} (2.4829)$$

$$+ \frac{(1)(-3)(-\frac{3}{2})}{(5)(1)(-2)} (2.4843) + \frac{(1)(-3)(-4)^2}{(7)(3)(2)} (2.4871)$$

$$= \frac{18}{35} \times 2.4771 + 2 \times 2.4829 + \frac{9}{-5} \times 2.4843$$

$$+ \frac{2}{7} \times 2.4871$$

$$= 1.273937 + 4.9658 - 4.47174 + 0.7105999$$

$$\log_{10} 301 = \underline{\underline{2.47859}}$$