Paper 7, TDC Part-3
Unit – 2, LU Decomposition Method

By:

Mayank Mausam

Assistant Professor (Guest Faculty)

Department of Electronics

L.S. College, BRA Bihar University,

Muzaffarpur, Bihar

Unit – 2, LU Decomposition Method

This is 3rd direct method to solve the system of given linear equations.

This method is also known as 'Factorization Method' or 'Crout's Method'

Let us look how this method works for finding the values of x, y, and z

Unit – 2, LU Decomposition Method

LU decomposition :→ (Factorization Or Crout's) Method.

⇒ Any square matrix can be expressed as the product of a lower triangular matrix and an upper triangular matrix

Consider the system of linear equation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{32}z = b_3$$

Unit – 2, LU Decomposition Method

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{32}z = b_3$$

In matrix form

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now let matrix $A = LU$

where L is a lower triangular matrix and is suppose to be

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Unit – 2, LU Decomposition Method

and U is upper triangular matrix and given by,

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$AX = B \Rightarrow LUX = B$$

Now let $UX = Y$ then,

$$LUX = B \Rightarrow LY = B$$

Unit – 2, LU Decomposition Method

As $A = LU$

So,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11} l_{21} & u_{12} l_{21} + u_{22} & u_{13} l_{21} + u_{23} \\ u_{11} l_{31} & u_{12} l_{31} + u_{22} l_{32} & u_{13} l_{31} + u_{23} l_{32} + u_{33} \end{bmatrix}$$

$u_{11} = a_{11}$, $u_{12} = a_{12}$ $u_{13} = a_{13}$

$l_{21} = \dfrac{a_{21}}{a_{11}}$, $u_{22} = a_{22} - a_{12} \times \dfrac{a_{21}}{a_{11}}$

$u_{23} = a_{23} - a_{13} \times \dfrac{a_{21}}{a_{11}}$

Unit – 2, LU Decomposition Method

$$l_{31} = \frac{a_{31}}{u_{11}}, \quad l_{32} = \frac{\left(a_{32} - \frac{a_{12} \times a_{31}}{u_{11}}\right)}{\left(a_{22} - \frac{a_{12} \times a_{21}}{a_{11}}\right)}$$

$$u_{33} = a_{33} - \frac{a_{13} \times a_{31}}{u_{11}} - u_{23}\, l_{32}$$

Once we obtain l & U matrix then we get $Y_{3\times1} = U_{3\times3} X_{3\times1}$ that results in 3×1 matrix.

Unit – 2, LU Decomposition Method

After that we find $L_{3\times3}Y_{3\times1}$ which is also 3×1 matrix

$$\begin{bmatrix} A_{11}\,x \\ A_{22}\,y \\ A_{31}\,Z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x = \frac{b_1}{A_{11}}, \quad y = \frac{b_2}{A_{21}}, \quad z = \frac{b_3}{A_{31}}$$

Now let us look at few examples.

Unit – 2, LU Decomposition Method

11) Solve the following system of linear equations by Factorization method.

$$x + 5y + z = 14$$
$$2x + y + 3z = 13$$
$$3x + y + 4z = 17$$

→ We can express given system of equation in matrix form as,

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$A \quad X = B$$

Unit – 2, LU Decomposition Method

or $AX = B$

Now let $A = LU$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

or $$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11} l_{21} & u_{12} l_{21} + u_{22} & u_{13} l_{21} + u_{23} \\ u_{11} l_{31} & u_{12} l_{31} + u_{22} l_{32} & u_{13} l_{31} + u_{23} l_{32} + u_{33} \end{bmatrix}$$

$u_{11} = 1$, $u_{12} = 5$, $u_{13} = 1$, $l_{21} = 2$,

$l_{31} = 3$

Unit – 2, LU Decomposition Method

$u_{22} = 1 - 10 = -9$, $u_{23} = 3 - 2 = +1$

$l_{32} = \dfrac{1 - 15}{-9} = \dfrac{+14}{+9}$, $u_{33} = 4 - 3 - \dfrac{14}{9} = \dfrac{-5}{9}$

So,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & \frac{-5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

Now, let

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & \frac{-5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 5y + z \\ -9y + z \\ \frac{-5z}{9} \end{bmatrix} = Y$$

Unit – 2, LU Decomposition Method

Now, $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} x+5y+z \\ -9y+z \\ -\frac{5z}{9} \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{x+5y+z} \\ \\ \\ \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} y_{11} \\ 2y_{11} + y_{21} \\ 3y_{11} + \frac{14}{9}y_{21} + y_{31} \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

Unit – 2, LU Decomposition Method

$y_{11} = 14 \quad , \quad y_{21} = 13 - 28 = -15$

$\cancel{y_{31} = 17 - 3\times 14 - \frac{14}{9} \times -1 = \frac{153 - 378 + 14}{9}}$

$\cancel{= \frac{167 - 378}{9} = \frac{-211}{9}}$

$y_{31} = 17 - 14\times 3 - \frac{14}{\cancel{9}_{3}} \times \cancel{-15}^{-5}$

$= 17 - 42 + \frac{70}{3} = \frac{-75 + 70}{3} = \frac{-5}{3}$

$$Y = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix} = UX = \begin{bmatrix} x + 5y + z \\ -9y + z \\ -\frac{5z}{9} \end{bmatrix}$$

Unit – 2, LU Decomposition Method

$$\begin{bmatrix} x + 5y + z \\ -9y + z \\ -\frac{5z}{9} \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

$$\Rightarrow \frac{-5z}{\cancel{9}_{3}} = \frac{-5}{\cancel{3}} \Rightarrow z = 3$$

then $-9y + 3 = -15 \Rightarrow y = \frac{+18}{+9} = 2$

$x + 5 \times 2 + 3 = 14$

$x = 14 - 13 = 1$

$x = 1, \quad y = 2, \quad z = 3$

Unit – 2, LU Decomposition Method

Problems for practice :-

Q1) Solve by Crout's Method

$$3x + y + z = 4$$
$$x + 2y + 2z = 3$$
$$2x + y + 3z = 4$$

Q2

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

Unit – 2, LU Decomposition Method

Thank You