Differential Equations of first order and higher degree:-

Case-II:- Equations solvable for y - If the given equation, on solving for y takes the form $y = f(x,p)$ ——(i)

Then differentiation with respect to x it gives as

$$p = \frac{dy}{dx} = g\left(x, p, \frac{dp}{dx}\right) \quad \left(\because \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\cdot\frac{dp}{dx}\right)$$

which is in variable p and x.

Now it may be possible to solve this new differential equation $g(x, p, \frac{dp}{dx})$ in x and p.

Let its solution be $F(x, p, c) = 0$ —— (ii)

The eliminating of p from (i) and (ii) gives the required solution.

In case elimination of p is not possible, then we may solve (i) & (ii) for x and y then obtain

$$x = F_1(p, c), \quad y = F_2(p, c)$$

where c is arbitrary constant.

Exp. Solve $y - 2px = \tan^{-1}(xp^2)$

Sol. Given Equation is

$$y = 2px + \tan^{-1}(xp^2) \quad \text{---(i)}$$

Differentiating both sides w.r.to x

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + \frac{1}{1+(xp^2)^2}\cdot\frac{d}{dx}(xp^2)$$

$$\frac{dy}{dx} = p = 2\left(p + x\frac{dp}{dx}\right) + \frac{p^2 + x\cdot 2p\frac{dp}{dx}}{1+x^2p^4}$$

$$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right)\left(\frac{p}{1+x^2p^4}\right) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)\left\{1 + \frac{p}{1+x^2p^4}\right\} = 0$$

This gives $p + 2x\frac{dp}{dx} = 0$

Separating the variables and integrating

$$\int\frac{dx}{x} + 2\int\frac{dp}{dp} = \log c \quad (c \text{ constant})$$

$$\log x + 2\log p = \log c$$

$$\log xp^2 = \log c$$

$$xp^2 = c \Rightarrow p = \sqrt{c/x} \quad \text{---(ii)}$$

Eliminating p from (i)

$$y = 2\left(\sqrt{c/x}\right)x + \tan^{-1}\left(x\left(\sqrt{c/x}\right)^2\right)$$

$$y = 2\sqrt{cx} + \tan^{-1}c$$ This is solution

Case-(iii)

Similarly, Equations solvable for x. If the given equation is the form $x = f(y,p) = 0$, then it is said to be solvable for x. In order to solve these types of differential equation we differentiate with respect to y we get

$$\frac{dx}{dy} = 1/p = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F\left(y, p, \frac{dp}{dy}\right)$$

which is variable p and y. Hence its solution is given by $F(y,p,c) = 0$.

The eliminating of p from $x = f(y,p)$ and $F(y,p,c) = 0$ gives the required solution.

Exp. Solve $y^2p^2 - 3xp + y = 0$ —(i)

Sol. Given equation is the form $x = f(y,p)$

$$x = \frac{1}{3}\left(\frac{y}{p} + y^2p\right) \quad \text{—(ii)}$$

differentiate with respect to y.

$$3\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2y\cdot p + y^2\frac{dp}{dy}$$

$$\frac{3}{p} = 2yp + \frac{1}{p} + \left(y^2 - \frac{y}{p^2}\right)\frac{dp}{dy} = 0$$

$$2yp - \frac{2}{p} + \frac{y}{p^2}(yp^2 - 1)\frac{dp}{dy} = 0$$

$$2p\left(y - \frac{1}{p^2}\right) + \frac{y}{p^2}(yp^2 - 1)\frac{dp}{dy} = 0$$

$$\frac{2}{p}(yp^2 - 1) + \frac{y}{p^2}(yp^2 - 1)\frac{dp}{dy} = 0$$

$$(yp^2 - 1)\left\{\frac{2}{p} + \frac{y}{p^2}\frac{dp}{dy}\right\} = 0$$

$(yp^2-1)\cdot\frac{1}{p^2}\left(2p+y\frac{dp}{dy}\right)=0$.

$$2p+y\frac{dp}{dy}=0$$

$$\frac{dp}{p}+2\frac{dy}{y}=0$$

Integrating both sides

$$\int\frac{dp}{p}+2\int\frac{dy}{y}=\log c$$

$$\log p+2\log y=\log c \quad (c \text{ a constant})$$

$$py^2=c$$

$$\Rightarrow p=\frac{c}{y^2}$$

Putting p value in Eqn (1)

$$y^2\left(\frac{c}{y^2}\right)^2-3x\frac{c}{y^2}+y=0$$

$$\frac{c^2}{y^2}-3x\frac{c}{y^2}+y=0$$

$$y^3-3xc+c^2=0$$

This is solution.