Theorem If $V(F)$ is a finite dimensional vector space (FDVS). Then a bilinear form f on V is symmetric or antisymmetric (skew-symmetric) according as the matrix of f in any ordered basis is symmetric or antisymmetric.

Proof: Let B be an ordered basis of FDVS $V(F)$. Let f be a bilinear form on V and $u, v \in V$.

Let A be the matrix representation of f relative to basis B. Also let X and Y be co-ordinate vectors of u, v relative to the basis B. Then, we have

$$f(u,v) = X'AY \quad \text{—— (i)}$$

$$f(v,u) = Y'AX \quad \text{—— (ii)}$$

As $X'AY$ is a 1×1 matrix [i.e X' is $1\times n$ matrix, A is $n\times n$ matrix and Y is $n\times1$ matrix], therefore

$$X'AY = (X'AY)' \qquad \because A = A' \text{ for sym.}$$

$$X'AY = Y'A'(X')' \qquad \because (AB)' = B'A'$$

$$X'AY = Y'A'X \quad \text{—— (iii)} \qquad (A')' = A$$

from (i), we get $f(u,v) = Y'A'X$

Case (i) if f is symmetrix then $f(u,v) = f(v,u)$

from (I) & (II) $\quad X'AY = Y'AX$ —— (iv)

from (III) & iv $\quad Y'AX = Y'A'X \quad \forall X, Y$

$$A = A'$$

$$\Rightarrow A \text{ is symmetric.}$$

Thus f is symmetric $\Rightarrow$ A is symmetric.

Case (ii) when f is anti-symmetric

Then $f(u,v) = -f(v,u)$ — v

from (i) & (ii)

$$X'AY = -Y'AX \quad \text{— (vi)}$$

from (iii) & (vi)

$$Y'A'X = -Y'AX$$

$$A' = -A \quad \text{or} \quad A = -A'$$

Hence f is antisymmetric $\Rightarrow$ A is anti-symmetric.

Finally, f is symmetric or anti-symmetric $\Rightarrow$ A is symmetric or anti-symmetric.