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For T.D.C. Part III

Paper-6

Gr—[redacted] A

The centre of a group. (1)

Definition. *The set Z of all self-conjugate elements of a group G is called the centre of G. Symbolically*

$$Z = \{z \in G : zx = xz \ \forall \ x \in G\}.$$

(Punjab 1968; B.H.U. 71; Meerut 90)

Theorem 4. *The centre Z of a group G is a normal subgroup of G.* **(Banaras 1971; Meerut 81, 90; Agra 86)**

Proof. We have $Z = \{z \in G : zx = xz \ \forall \ x \in G\}$.

First we shall prove that Z is a subgroup of G.

Let $z_1, z_2 \in Z$. Then $z_1 x = x z_1$ and $z_2 x = x z_2$ for all $x \in G$.

We have $z_2 x = x z_2 \ \forall \ x \in G$

$$\Rightarrow z_2^{-1} (z_2 x) z_2^{-1} = z_2^{-1} (x z_2) z_2^{-1}$$

$$\Rightarrow x z_2^{-1} = z_2^{-1} x \ \forall \ x \in G$$

$$\Rightarrow z_2^{-1} \in Z.$$

Now $(z_1 z_2^{-1}) x = z_1 (z_2^{-1} x) = z_1 (x z_2^{-1}) = (z_1 x) z_2^{-1} = (x z_1) z_2^{-1}$

$$= x (z_1 z_2^{-1}).$$

$\therefore \ z_1 z_2^{-1} \in Z.$

Thus $z_1, z_2 \in Z \Rightarrow z_1 z_2^{-1} \in Z$.

$\therefore$ Z is a subgroup of G.

(2)

Now we shall show that Z is a normal subgroup of G. Let $x \in G$ and $z \in Z$. Then

$$xzx^{-1}=(xz)\,x^{-1}=(zx)\,x^{-1}=z \in Z.$$

Thus $$x \in G, z \in Z \Rightarrow xzx^{-1} \in Z.$$

$\therefore$ Z is a normal subgroup of G.

Give an example to show that in a group G the normaliser of an element is not necessarily a normal subgroup of G.

(Meerut 1985, 91; B.H.U. 88)

Solution. Consider the group S_3, the symmetric group of permutations on three symbols a, b, c. We have $S_3=\{I, (ab), (bc), (ca), (abc), (acb)\}$. Let $N(ab)$ denote the normaliser of the element $(ab) \in S_3$. We shall show that $N(ab)$ is not a normal subgroup of S_3. Let us calculate the elements of $N(ab)$. Obviously $(ab) \in N(ab)$. Also $I \in N(ab)$ because $I(ab)=(ab)I$

Now $(bc)(ab)=(abc)$ and $(ab)(bc)=(acb)$. Thus (bc) does not commute with (ab). Therefore $(bc) \notin N(ab)$. Again

$$(ca)(ab)=(acb) \text{ and } (ab)(ca)=(abc).$$

Thus $(ca)(ab) \neq (ab)(ca)$ and therefore $(ca) \notin N(ab)$. Similarly we can verify that $(abc) \notin N(ab)$ and $(acb) \notin N(ab)$. Hence

$$N(ab)=\{I, (ab)\}.$$

Now we shall show that $N(ab)$ is not a normal subgroup of S_3. Take the element $(bc) \in S_3$ and the element $(ab) \in N(ab)$. We have $(bc)(ab)(bc)^{-1}=(bc)(ab)(cb)=(abc)(cb)=(ac) \notin N(ab)$. Therefore $N(ab)$ is not a normal subgroup of S_3.

Q. (5) *Let Z denote the centre of a group G. If G|Z is cyclic prove that G is abelian.* (Meerut 1978, 81; I.C.S. 90; Guru Nanak 89, Madurai 88)

Solution It is given that G/Z is cyclic. Let Zg be a generator of the cyclic group G/Z where g is some element of G.

Let $a, b \in G$. Then to prove that $ab=ba$. Since $a \in G$, therefore $Za \in G/Z$. But G/Z is cyclic having Zg as a generator. Therefore there exists some integer m such that $Za=(Zg)^m=Zg^m$, because Z is a normal subgroup of G. Now $a \in Za$. Therefore

$$Za=Zg^m \Rightarrow a \in Zg^m \Rightarrow a=z_1 g^m \text{ for some } z_1 \in Z.$$

Similarly $b=z_2 g^n$ where $z_2 \in Z$ and n is some integer.

Now $ab=(z_1 g^m)(z_2 g^n)=z_1 g^m z_2 g^n$

$=z_1 z_2 g^m g^n$ $[\because \ z_2 \in Z \Rightarrow z_2 g^m=g^m z_2]$

$=z_1 z_2 g^{m+n}$.

Again $ba=z_2 g^n z_1 g^m=z_2 z_1 g^n g^m=z_2 z_1 g^{n+m}$

$=z_1 z_2 g^{m+n}$ $[\because \ z_1 \in Z \Rightarrow z_1 z_2=z_2 z_1]$

$\therefore \ ab=ba.$

Since $ab=ba \ \forall \ a, b \in G$, therefore G is abelian.

Normalizer of an element of a group.

Definition. *If $a \in G$, then $N(a)$, the normalizer of a in G is the set of all those elements of G which commute with a. Symbolically $N(a)=\{x \in G : ax=xa\}$.*

(I.A.S. 1975; Nagarjuna 78; Meerut 81, 84, 88; B.H.U. 88)

Theorem 2. *The normalizer $N(a)$ of $a \in G$ is a subgroup of G.*

(Agra 1986; I.A.S. 72; Kanpur 86; Meerut 84, 88; Punjab 70)

Proof. We have $N(a)=\{x \in G : ax=xa\}$.

Let $x_1, x_2 \in N(a)$. Then $ax_1=x_1a$, $ax_2=x_2a$.

First we show that $x_2^{-1} \in N(a)$.

We have $ax_2=x_2a \Rightarrow x_2^{-1}(ax_2)x_2^{-1}=x_2^{-1}(x_2a)x_2^{-1}$

$$\Rightarrow x_2^{-1}a=ax_2^{-1} \Rightarrow x_2^{-1} \in N(a).$$

Now we shall show that $x_1x_2^{-1} \in N(a)$.

We have $a(x_1x_2^{-1})=(ax_1)x_2^{-1}=(x_1a)x_2^{-1}$

$$=x_1(ax_2^{-1})=x_1(x_2^{-1}a)=(x_1x_2^{-1})a.$$

$\therefore$ $x_1x_2^{-1} \in N(a)$.

Thus $x_1, x_2 \in N(a) \Rightarrow x_1x_2^{-1} \in N(a)$

$\therefore$ $N(a)$ is a subgroup of G.

Conjugate elements.

Definition. (Agra 1969; Banaras 61)

If a, b be two elements of a group G, then b is said to be conjugate to a if there exists an element $x \in G$ such that

$$b = x^{-1} ax.$$

If $b = x^{-1} ax$, then b is also called the **transform** of a by x.

If b is conjugate to a then symbolically we shall write $b \sim a$ and this relation in G will be called the *relation of conjugacy*. Thus $b \sim a$ iff $b = x^{-1} ax$ for some $x \in G$.

Theorem 1. *The relation of conjugacy is an equivalence relation on G.* (Vikram 1976; Banaras 61; Kanpur 88)

Proof. Reflexivity. If a is any element of G, then we have

$$a = e^{-1} ae \Rightarrow a \sim a.$$

Thus $a \sim a \; \forall \; a \in G$. Therefore the relation is reflexive.

Symmetry. We have $a \sim b \Rightarrow a = x^{-1} bx$ for some $x \in G$ $\Rightarrow xax^{-1} = x(x^{-1} bx)x^{-1} \Rightarrow xax^{-1} = b \Rightarrow b = (x^{-1})^{-1} ax^{-1}$ where $x^{-1} \in G$ $\Rightarrow b \sim a$.

Therefore the relation is symmetric.

Transitivity. Let $a \sim b, b \sim c$. Then $a = x^{-1}bx, b = y^{-1}cy$ for some $x, y \in G$. From this we get

$$a = x^{-1}(y^{-1} cy) x \qquad [\because b = y^{-1} cy]$$

$$= (yx)^{-1} c (yx) \text{ where } yx \in G.$$

$\therefore a \sim c$ and thus the relation is transitive. Hence the relation of conjugacy in a group G is an equivalence relation. Therefore it will partition G into disjoint equivalence classes called **classes of conjugate elements.** These classes will be such that

(i) any two elements of the same class are conjugate.

(ii) no two elements of different classes are conjugate.

The collection of all elements conjugate to an element $a \in G$ will be symbolically denoted by $C(a)$ or by $\bar{a}$. Thus

$$C(a)=\{x \in G : x \sim a\}.$$

$C(a)$ will be called the *conjugate class of a in* G. We have $(y^{-1}ay) \sim a$ for all $y \in G$. Also if $b \sim a$ then b must be equal to $y^{-1}ay$ for some $y \in G$. Therefore $C(a)=\{y^{-1}ay : y \in G\}$.

If G is a finite group, then the number of distinct elements in $C(a)$ will be denoted by c_a.