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For T.D.C. Part II
Paper – 3

Abstract (Modern) Algebra

► **Ex.6.** *Show that a group with 4 or fewer elements is necessarily Abelian.*

Soln. A group with only one element consists of the identity element e alone and e... so the group is Abelian.

A group with two elements consists of e (the identity element) and another elem... $a \neq e$. But $ae = ea$.

Hence the group is Abelian.

Let G be a group with three elements, say e, a, b. If a and b are their own inverse... since e is also its own inverse, then by the preceding example G is Abelian. On the other... if $a = b^{-1}$, then $ab = b^{-1}b = e$ and $ba = bb^{-1} = e$ so that $ab = ba$.

Also, clearly $ae = ea$ and $be = eb$.

Hence G is Abelian.

Let $G = \{e, a, b, c\}$ be a group of order 4. Here e is the identity element. The ide... element e is its own inverse. Leaving aside e, we have got three elements a, b, c. Hence... must be at least one more element in G which is its own inverse.

Let $a^{-1} = a$.

We have now the following cases.

(i) If $b^{-1} = b$ and $c^{-1} = c$, then G must be necessarily Abelian, for the following rea... (vide Ex. 4).

$$b, c \in G \Rightarrow bc \in G$$
$$\Rightarrow (bc)^{-1} \in G$$
$$\Rightarrow (bc)^{-1} = bc; \text{ since every element is its own inverse}$$
$$\Rightarrow c^{-1}b^{-1} = bc$$
$$\Rightarrow cb = bc$$
$$\Rightarrow G \text{ is Abelian.}$$

If H_1 and H_2 be two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G

Proof : Let x_1 and x_2 both $\in H_1 \cap H_2$.

Then $x_1, x_2 \in H_1$ and also $x_1, x_2 \in H_2$.

Therefore $x_1 x_2 \in H_1$ and $x_1 x_2 \in H_2$

Hence $x_1 x_2 \in H_1 \cap H_2$

Again $x_1 \in H_1$ and $x_1 \in H_2$

$\therefore x_1^{-1} \in H_1$ and $x_1^{-1} \in H_2$.

Hence $x_1^{-1} \in H_1 \cap H_2$.

Thus in accordance with the theorem 1 of 2.2, $H_1 \cap H_2$ is a subgroup of G.

(i) We are given that for all $a, b \in H, ab \in H$ and hence the first postulate is satisfied.

(ii) H is associative because H is a subset of G which is associative.

(iii) If $x \in H$, then because of condition (ii), $x^{-1} \in H$ and the fourth postulate is satisfied.

(iv) Again, we have by (i), $xx^{-1} \in H$, i.e., $e \in H$. Hence the identity element of H is necessarily e.

Thus, we see that H satisfies all the four postulates of a group and hence it is a subgroup.

The conditions (i) and (ii) of this theorem can be replaced by the following single condition.

D.A

Acco.