Dr. O. P. Raman
Deptt. of Mathematics

For T.D.C. Part III
Paper-5
Gr- A

State & prove Inverse Function Theorem

Statement:- If the continuous function $f(x)$ be strictly increasing (or decreasing) in the closed interval $[a, b]$, the equation $y = f(x)$ defines an inverse function $x = \phi(y)$ which is also continuous and strictly increasing (or decreasing) in the interval $f(a) \le y \le f(b)$

$y = f(x)$
$\therefore x = f^{-1}(y)$
i.e. $x = \phi(y)$

Proof:- Let the function $y = f(x)$ be continuous and (say) strictly increasing in $a \le x \le b$. Then if x_1, x_2 lie in this interval,

$x_2 > x_1$ implies $f(x_2) > f(x_1)$ — (1)

and conversely

Let $f(a) = \alpha$ and $f(b) = \beta$ and y be any number between α & β. Then for some value x_1 of x between a & b, we must have $f(x_1) = y_1$. Moreover this value is unique, for

$f(x_1) = f(x_2)$ implies $x_1 = x_2$ — (2)

Thus the relation $y = f(x)$ sets up one-to-one correspondence between x & y denoted by $x \longleftrightarrow y$ over the intervals $a \le x \le b$ and $\alpha \le y \le \beta$.

Hence x is an inverse function of y over $\alpha \le y \le \beta$ and if we denote this inverse function by $x = \phi(y)$, we have $y = f\{\phi(y)\}$, $x = \phi\{f(x)\}$

$y = f(x)$
$= f\{\phi(y)\}$
& $x = \phi(y) = \phi\{f(x)\}$

If $x_1 = \phi(y_1)$, $x_2 = \phi(y_2)$, the converse of (1) states that

$y_2 > y_1$ implies $\phi(y_2) > \phi(y_1)$.

i.e. $\phi(y)$ is also a strictly increasing function in $\alpha \le y \le \beta$.

Next, to prove the continuity of $\phi(y)$ in this interval, we consider the positive function

$F(x) = f(x + \epsilon) - f(x)$, ϵ being an arbitrary small positive quantity.

We take $F(x)$ as continuous function in $a \le x \le b$.

Since $F(x)$ is continuous in $a \le x \le b - \epsilon$, it assumes its least value δ at some point of this interval, hence if x_1, x_2 are two points for which $|x_1 - x_2| \geq \epsilon$, then since $f(x)$ is strictly increasing,

$|f(x_1) - f(x_2)| = |y_1 - y_1| \geq \delta$.

Consequently when $|y_1 - y_2| < \delta$, we must have

$|x_1 - x_2| = |\phi(y_1) - \phi(y_2)| < \epsilon$

i.e. $x = \phi(y)$ is continuous throughout the interval.

Q. Obtain a set of sufficient conditions under which a function $f(x, y)$ is differentiable at a given point (a, b).

Here we propose to show that if a function $f(x, y)$ defined in a certain neighbourhood V of a point (a, b) be such that

(i) $f_x(a, b)$ and $f_y(a, b)$ exist

(ii) one of $f_x(x, y)$ and $f_y(x, y)$ is continuous at (a, b),

then $f(x, y)$ is differentiable at (a, b)

Proof:- Let $(a+h, b+k)$ be any point in the neighbourhood V of (a, b) in which f_x & f_y exist.

Let $f_y(x, y)$ is continuous at (a, b).

Now, we have

$$f(a+h, b+k) - f(a, b) = f(a+h, b+k) - f(a+h, b) + f(a+h, b) - f(a, b) \quad \text{———} (1)$$

Since f_y exists in V, by Mean value theorem we have

$$f(a+h, b+k) - f(a+h, b) = k \cdot f_y(a+h, b+\theta k) \quad \text{——} (2)$$

where θ lies between 0 & 1

[Mean value theorem is $f(a+h) - f(a) = hf'(a+\theta h)$, where $0<\theta<1$]

Also, since $f_y(x,y)$ is continuous at (a,b), we have

$$\lim_{\substack{x\to a\\ y\to b}} f_y(x,y) = f_y(a,b)$$

$$\therefore \lim_{\substack{h\to 0\\ k\to 0}} f_y(a+h, b+\theta k) = f_y(a,b) \quad [\text{putting } x = a+h \;\&\; y = b+\theta k]$$

$$\therefore f_y(a+h, b+\theta k) = f_y(a,b) + \psi(h,k), \quad \text{——} (3)$$

where $\psi(h,k) \to 0$ as $(h,k) \to 0$

Again, since $f_x(a,b)$ exists, therefore by definition of $f_x(a,b)$,

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h} = f_x(a,b)$$

$$\therefore f(a+h, b) - f(a,b) = h\left[f_x(a,b) + \phi(h)\right], \quad \text{——} (4)$$

where $\phi(h) \to 0$ as $h \to 0$

Using (2) & (4) in (1), we get

$$f(a+h, b+k) - f(a,b) = k f_y(a+h, b+\theta k) + h\left[f_x(a,b) + \phi(h)\right],$$

where $\phi(h) \to 0$ as $h\to 0$

Using (3), we get

$$f(a+h, b+k) - f(a,b) = k\left[f_y(a,b) + \psi(h,k)\right] + h\left[f_x(a,b) + \phi(h)\right],$$

where $\lim_{h\to 0} \phi(h) = \lim_{\substack{h\to 0\\ k\to 0}} \psi(h,k) = 0$

i.e. $$f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\phi(h) + k\psi(h,k), \quad \text{———} (5)$$

where $\lim_{h\to 0} \phi(h) = \lim_{\substack{h\to 0\\ k\to 0}} \psi(h,k) = 0$

But (5) means that the function $f(x,y)$ is differentiable at (a,b) by definition. So its completes the first part of the theorem.