CONSTRAINTS.

The limitations or geometrical restrictions on the motion of a particle or system of particles are generally known as constraints.

A particle needs three independent parameters (degree of freedom) to specify its position in space. If the particle is assumed to be moving on table top, its motion is confined to the surface of the table top only and it requires two parameters to locate its position on the top. In this case its own motion is said to be restricted or limited. Then, it is said to be under constraint. Thus the constraints reduce the number of independent coordinates (degrees of freedom). Similarly, in the motion of a particle moving on a st. line, there are two constraint present.

Classification of Constraints.

If the conditions of constraint are expressed as equations connecting the coordinates of particle having the form

$$f(\vec{r}_1, \vec{r}_2 \cdots\cdots, t) = 0$$

then the constraint are said to be holonomic. As for example, the constraints of a rigid body may be given by

$$(\vec{r}_i - \vec{r}_j)^2 - C_{ij}^2 = 0$$

For another example the motion of point mass of a simple pendulum is restricted, since the distance between the point mass and point of suspension is constant. The condition of constraint are expressed as

$(\vec{r} - \vec{a})^2 = l^2$ where $\vec{r}$ is the position vector of the point mass & $\vec{a}$ is the position vector of point of suspension.

The constraints which can't be expressed in this fashion are called <u>Non-holonomic</u>. As for example, the motion of the gas molecules within the Container is restricted by the walls of the vessel. Another example, the motion of a Particle placed on the Surface of sphere, which may be expressed as $r^2 - a^2 \geq 0$, where a is the radius of the sphere.

If the Constraints are independent of time, they are termed as Schleronomic, but if they Contain time explicitly, they are Called Rheonomic. A bead sliding on a moving wire is an example of Rheonomic Constraints.

Generalised Co-ordinates

Any set of independent Co-ordinates (or variables) sufficient in number to define unambiguously the system Configuration is Called generalised Coordinate.

These generalised Coordinates may Consist of Cartesian Coordinates, Polar Co-ordinates, Spherical Polar Coordinates. These are generally denoted by $q_1, q_2, q_3 \ldots q_f$, f refers to the number of degrees of freedom.

To define the Position of a System of N- Particles in space, 3N independent Coordinates or degrees of freedom are Required. If Constraints are expressed by p equations, then these equations may be used to eliminate p of the 3N Coordinates thereby leading 3N−p independent Coordinates with 3N−p degrees of freedom of the system. Thus we have 3N−p independent variables $q_1, q_2 \ldots q_{3N-p}$ into terms of which the old coordinates $r_1, r_2 \ldots r_n$ Can be expressed as

$$\vec{r}_1 = \vec{r}_1 (q_1, q_2 \ldots q_{3N-p}, t)$$
$$\vec{r}_2 = \vec{r}_2 (q_1, q_2 \ldots q_{3N-p}, t)$$
$$\vec{r}_N = \vec{r}_N (q_1, q_2 \ldots q_{3N-p}, t)$$

Here $q_1, q_2 \dots q_{3N-p}$ are called generalised coordinates of the Particle. The number $K = 3N-p$ is called the no. of degrees of freedom of the system.

The Configuration ~~space~~ of the system may be represented by the Position of a Point in a K-dimensional space which is called the Configuration space of the system.

The generalised Coordinates of the function of N-variables $\vec{r}_1, \vec{r}_2 \dots \vec{r}_N$ and one time variable t, are given by

$$q_i = q_i(\vec{r}_1, \vec{r}_2, \dots r_N, t)$$

where $i = 1, 2 \dots K$,

They are independent of each other. By use of the generalised coordinates the general 3N dimensional Configuration ~~spa~~ is reduced to 3N-p dimensional space & get rid of forces due to Constraints.

Elimination of Constraints in this manner reduces the number of Coordinates to a minimum.

D' Alembert's Principle

Let any no. of forces be applied to the i-th Particle of a system at any instant of time. The virtual displacement of a system refers to a change in the Configuration of the system. If the system is in equilibrium then the total force on each Particle is zero i.e. $F_i = 0$

So $\vec{F}_i \cdot \delta\vec{r}_i = 0$,

where $\delta\vec{r}_i$ is virtual displacement and F_i is the force in equilibrium.

on the ith Particle. Since sum of these vanishing. Product over all Particles must likewise be zero, therefore

$$\sum \vec{F_i} \cdot \delta \vec{r_i} = 0.$$

This means that if system of free Particles is equilibrium, the total work done by all forces, internal and external, in a virtual displacement is zero.

The force $\vec{F_i}$ is written as

$$\vec{F_i} = \vec{F_i}^a + \vec{f_i}, \quad \vec{f_i} \text{ being force of}$$

Constraint. Therefore the above equation becomes

$$\sum_i \vec{F_i}^a \cdot \delta \vec{r_i} + \sum_i \vec{f_i} \cdot \delta \vec{r_i} = 0$$

If we take force of Constraints normal to motion then. $\vec{f_i} \cdot \delta \vec{r_i} = 0$ since $\delta \vec{r_i} \perp \vec{f_i}$.

Then we have $\sum \vec{F_i}^a \cdot \delta \vec{r_i} = 0$.

Thus for equilibrium of a system virtual work of the applied force vanishes. This is known as Principle of virtual work.

This Principle was modified into a new Principle by J. Bernouli & developed by D' Alembert by taking

$$\vec{F_i} = \dot{\vec{P_i}} \quad \text{Then} \quad \vec{F_i} - \dot{\vec{P_i}} = 0.$$

which states that the Particle in the system will be in equilibrium under a force equal to the actual force plus a reversed effective force $-\dot{P_i}$. Thus the above equation takes the form

$$\sum_i (\vec{F_i} - \dot{\vec{P_i}}) \cdot \delta \vec{r_i} = 0$$

or $$\sum_i (\vec{F_i}^a - \dot{P_i}) \cdot \delta \vec{r_i} + \sum_i \vec{f_i} \cdot \delta \vec{r_i} = 0$$

If the virtual work of forces of constraint vanishes, then

$$\sum_i (\vec{F}_i^{a} - \dot{\vec{P}}_i) \cdot \delta \vec{r}_i = 0$$

which is known as D'Alembert's Principle where the forces due to constraint disappear and we may use $\vec{F}_i$ in place of $\vec{F}_i^{a}$ without ambiguity.

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Lagrange eqn of Motion

Any set of independent co-ordinates (or variables) sufficient in no. to define unambiguously the system configuration is called generalised co-ordinates.

The generalised co-ordinates of the fn. of N variables $\vec{r}_1, \vec{r}_2 \dots \vec{r}_N$ and one time t is given by

$$q_i = q_i(\vec{r}_1, \vec{r}_2 \dots \vec{r}_N, t)$$

& the variables are expressed as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots q_{3N-k}, t)$$
$$= \vec{r}_i(q_1, q_2 \dots q_f, t)$$

where $q_1, q_2, q_3 \dots q_f$ are called generalised co-ordinates.

Let us consider a system of particles having f independent generalised co-ordinates to specify the state of its particles. The transformation equations of the generalised co-ordinates are given by

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3 \dots q_f, t) \longrightarrow (1)$$

Let us now suppose that any particle (i th) of mass m_i be acted upon by an external force $\vec{F}_i$. Then according to d' Alembert's Principle, we have

$$\sum(\vec{F}_i - \dot{\vec{P}}_i) \cdot \delta\vec{r}_i = 0$$

$$\text{or} \quad \sum(\vec{F}_i - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0 \qquad (2)$$

where $-\dot{\vec{P}}_i$ is the reverse effective force & $\delta\vec{r}_i$ is the virtual displacement of i th particle due to external force $\vec{F}_i$.

Now, applying the usual rules of Calculus of Partial differentiation of eqn (1), we obtain

$$\dot{\underline{r}}_i = \frac{\delta \underline{r}_i}{\delta t} = \underline{v}_i = \frac{\partial \underline{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \underline{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \underline{r}_i}{\partial t}$$

$$= \sum_k \frac{\partial \underline{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \underline{r}_i}{\partial t} \quad \text{—— (3)}$$

Similarly, the arbitrary virtual displacement $\delta \underline{r}_i$ can be connected with virtual displacement in terms of generalised co-ordinates δq_k by

$$\delta \underline{r}_i = \sum_k \frac{\partial \underline{r}_i}{\partial q_k}\delta q_k \quad \text{—— (4)}$$

Since there is no variation of time δt in the above eqn because virtual displacement takes no variation of time.

Now eqn (2) can now be written in the form as

$$\sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum \underline{F}_i \cdot \delta \underline{r}_i \quad \text{—— (5)}$$

Putting the value of eqn (4) in (5), we get

$$\sum_i \sum_k m_i \ddot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_k}\delta q_k = \sum_i \sum_k \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_k}\delta q_k \quad \text{—— (6)}$$

Let us now consider the relation

$$\sum_i m_i \ddot{\underline{r}}_i \frac{\partial r_i}{\partial q_k} = \sum_i \left\{ \frac{d}{dt}\left(m_i \dot{\underline{r}}_i \frac{\partial r_i}{\partial q_k}\right) - m_i \dot{\underline{r}}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) \right\}$$

Dr. C

Since $\frac{d}{dt}\left(\ddot{r}_i \frac{\partial r_i}{\partial q_k}\right) = \ddot{r}_i \frac{\partial r_i}{\partial q_k} + \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)$

$$\therefore \ddot{r}_i \frac{\partial r_i}{\partial q_k} = \frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_k}\right) - \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)$$

with use of the above eqn. in eqn (6), we get

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_k}\right) - \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right)\right] \delta q_k$$

$$= \sum_i \sum_k F_i \frac{\partial r_i}{\partial q_k} \delta q_k \quad \text{— (7)}$$

Differentiating eqn (5) w.r.t. $\dot{q}_k$ partially

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_k} = \frac{\partial r_i}{\partial q_k} \quad \text{— (8)}$$

Again differentiating eq. (5) w.r.t q_k partially

$$\frac{\partial \dot{r}_i}{\partial q_k} = \frac{\partial^2 r_i}{\partial q_k \partial q_k} \dot{q}_1 + \dots + \frac{\partial^2 r_i}{\partial t \, \partial q_k}$$

$$= \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial t \, \partial q_k} \quad \text{— (9)}$$

which is analogous to

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial r_i}{\partial q_k}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial r_i}{\partial q_k}\right)\dot{q}_2 + \cdots$$

$$+ \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_k}\right)$$

$$= \sum_k \frac{\partial^2 r_i}{\partial q_k \partial q_k}\dot{q}_k + \frac{\partial^2 r_i}{\partial t \partial q_k} \quad \text{— (10)}$$

Therefore we have on comparing eq (9) & (10)

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial \dot{r}_i}{\partial q_k} \quad \text{— (11)}$$

We have then

$$\frac{d}{dt}\left(\dot{r}_i \frac{\partial r_i}{\partial q_k}\right) = \frac{d}{dt}\left(\dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_k}\right) \quad \text{from eqn (8)}$$

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{r}_i^2\right)\right\} \quad \text{— (12)}$$

Making substitution of eq. (11) & (12) in (7) we get —

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$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{r}_i^2\right)\right\} - \dot{r}_i \frac{\partial \dot{r}_i}{\partial q_k} \right] \delta q_k = \sum_i \sum_k F_i \frac{\partial r_i}{\partial q_k} \delta q_k$$

$$\Rightarrow \sum_i \sum_k \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} m_i \dot{r}_i^2\right)\right\} - \frac{\partial}{\partial q_k}\left(\frac{1}{2} m_i \dot{r}_i^2\right) \right] \delta q_k = \text{———}$$

$$\Rightarrow \sum_k \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial \dot{q}_k}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right)\right\} - \frac{\partial}{\partial q_k}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right) \right] \delta q_k = \text{———}$$

T = Total K.E & $\sum_i F_i \frac{\partial r_i}{\partial q_k} = Q_k$ is the component of generalized force

$$\sum_k \left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right\} \delta q_k = \sum_k Q_k \delta q_k \text{ ———}$$

$$\text{or } \sum_k \left\{ \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right) - Q_k \right\} \delta q_k = 0 \text{ ——— } 14$$

In order to hold the eqn good, the co-eff of δq_k must vanish

i.e. $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} - Q_k = 0$

$\therefore \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$ ——— (15)

General form of Lagrange's eq.

when the system is wholly conservative

For conservative system the force is given by

$\bar{F}_i = -\nabla V_i = \frac{-\partial V_i}{\partial r_i}$

generalized force

$$Q_k = \sum_i F_i \frac{\partial r_i}{\partial q_k} = -\sum_i \frac{\partial V_i}{\partial r_i} \frac{\partial r_i}{\partial q_k} = \sum -\frac{\partial V_i}{\partial q_k} \Rightarrow -\frac{\partial V}{\partial q_k}$$

Hence

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} = -$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial}{\partial q_k}(T-V) = 0$$

$$\frac{d}{dt}\left\{\frac{\partial (T-V)}{\partial \dot{q}_k}\right\} - \frac{\partial}{\partial q_k}(T-V) = 0$$

putting $L = T - V$ = Lagrangian fn

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0} \quad (16)$$

~~we may use~~

This is the required Lagrangian eqn of motion for conservative system

/

HAMILTON'S CANNONICAL EQUATION OF MOTION

In the formulation of the laws of mechanics with the help of Lagrangian and the equation of motion follows from it, it has been presumed that the mechanical state of the system is completely defined. There is yet another mode of description, much more powerful & advantageous than the Lagrangian approach in terms of generalised co-ordinates and velocities. In this formulation the state is described in terms of generalised co-ordinates and momenta. The natural eqn which now arises is another approach of mechanics. For getting this new approach, let us recall the Lagrangian of the system as given by

$$L = L(q_k, \dot{q}_k, t) \quad \text{---} ①$$

and so $$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t} \quad \text{--} ②$$

and the Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

or $$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \quad \text{-----} ③$$

with this substitution, eqn ② becomes

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) + \frac{\partial L}{\partial t}$$

ie $$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right) = \frac{d}{dt}\left\{L - \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right\} \text{—} ④$$

But $\frac{\partial L}{\partial \dot{q}_k} = P_k$ = generalised momenta

Thus eqn (4) may be written as

$$\frac{d}{dt}\left\{L - \sum_k P_k \dot{q}_k\right\} = \frac{\partial L}{\partial t} \quad \text{or} \quad \frac{d}{dt}\left\{\sum_k P_k \dot{q}_k - L\right\} = -\frac{\partial L}{\partial t} \quad \text{---} (5)$$

In this eqn $\sum_k P_k \dot{q}_k - L$ is the integral and corresponds to the energy of the system given as $\sum_k P_k \dot{q}_k - L = H$

where H is known as Hamiltonian function, or

$$H = \sum_k P_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad \text{----} \quad (6)$$

The nature of new fn H still remains to be determined. To achieve this we have from eqn (6) that there are no terms in the right which involve the variation in the velocities $\dot{q}_k$. Hence it is seen that H may be written as

$$H = \sum_k P_k \dot{q}_k - L(q_k, \dot{q}_k, t) = H(P_k, q_k, t) \quad \text{---} (7)$$

Which is called the Hamiltonian.

The differential of H is given by

$$dH = \sum_k \frac{\partial H}{\partial P_k} dP_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \quad \text{——} (8)$$

But from eqn (6)

$$dH = \sum_k \dot{q}_k \, dP_k + \sum_k P_k d\dot{q}_k - dL \quad \text{---} (9)$$

Since $L = L(q_k, \dot{q}_k, t)$

$$\therefore \, dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

Hence, $dH = \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt$

where $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $\dot{p}_k = \frac{\partial L}{\partial q_k}$

$$= \sum_k \dot{q}_k dp_k + \sum_k p_k d\dot{q}_k - \sum_k \dot{p}_k dq_k - \sum_k p_k d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \sum_k \dot{q}_k dp_k - \sum_k \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \quad \text{--- (10)}$$

Now comparing the co-efficients of dp_k, dq_k and dt in eqn. (8) and eqn. (10), we get

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{-- (11)}$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} \quad \text{— (12)}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \text{— (13)}$$

Eqn (11) & (12) are called Hamilton's eqn or Hamilton's canonical eqn of motion. These are the required equations of motion in the variables p & q. The variables p_k & q_k are said to be canonically conjugate. Hamilton's equation constitute 1st 6N order differential eqn for a system of N particles in this rectangular co-ordinates in place of 3N Lagrangian eqn of motion of 2nd order.

PHYSICAL SIGNIFICANCE OF Hamiltonian (H)

Since $H = H(p_k, q_k, t)$

Hence $\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t}$

$$= -\sum_k \dot{p}_k \dot{q}_k + \sum_k \dot{p}_k \dot{q}_k + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

But $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, Hence $\frac{dH}{dt}$

~~But ∂H~~ If L is not an explicit fn of time,

$\frac{\partial L}{\partial t} = 0$, thereby giving $\frac{dH}{dt} = 0$ so H = Constant

Thus we may show that if the Lagrangian L is not an explicit fn of time, the Hamiltonian H is constant of motion. This means that it will not appear in H.

For Conservative system the P.E V is coordinate dependent but velocity independent i.e. $\frac{\partial V}{\partial \dot{q}_k} = 0$

Also $H = \sum_k p_k \dot{q}_k - L = 2T - L = 2T - T + V$

$= T + V = E =$ Total energy of the System

Thus for Conservative system where the Co-ordinate transformation is independent of time, the Hamiltonian function H represents the total energy of the System.

Thus H possesses the dimension of energy but in all Cases it is not equal to total energy E.

Advantages: Hamiltonian approach is advantageous than Lagrangian approach. Because of the fact that Hamiltonian approach gives deeper insight of the Physical Problems than Lagrangian. In another way we can say that in case of Hamiltonian approach we have to solve two differential equations of first order than one 2nd order differential eqn in Lagrangian. It is easier to solve first order d.E than to 2nd order d.E. Hence it is superior to Lagrangian approach.

Frame of Reference

Experiments as well as simple observations reveal that the motion is only relative. Absolute motion has no physical meaning. At best the motion of one body is described ~~to~~ relative to any other well defined system or body. This well defined system relative to which the motion of an object described is called a "frame of reference." For instance, the motion of a flying aircraft is specified w.r.to the coordinate system fixed on the earth; the motion of a charged particle in a particle accelerator is given relative to the accelerator. Here earth and accelerator are the frames of reference.

Broadly, there are two kinds of frame of references: (a) Inertial frame of reference & (b) Non-inertial frame of reference.

Inertial frame of reference

The frame relative to which the body is either at rest or moving with uniform, linear velocity ~~are~~ is known as inertial frame of reference.

Or Two frames can be said to be of inertial frames of reference with respect to one another when they are either at rest or in uniform relative motion with respect to one another.

In such frames, space is homogeneous and isotropic and time is also homogeneous. In particular, in such a frame a free body at rest at any instant will remain always at rest or a body moves with constant velocity in absence of any external force.

Thus, we see that an inertial frame is one in which Law of inertia or Newton's first Law of motion is valid.

Another Property ~~for de~~ that can be utilized for defining inertial frames is the one according to which the equation of motion of a body takes on the simplest form.

The equations of motion of a Particle are invariant under Galilean transformation since they Preserve their form when transformed from one inertial frame to another inertial frame moving with uniform velocity.

Non-Inertial frame of reference.

The frames relative to which the body not acted upon by external force, is accelerated are called non-inertial frames.

It clearly shows that newton's Law of motion does not hold true. From experimental inferences, we know that any thing capable of rotating must have acceleration even in absence of external force. On this basis we can say that the earth is not an inertial frame. Because earth is rotating. Hence any system fixed in earth is also non-inertial frame.

It can be easily shown that any frame moving with constant angular velocity relative to any inertial frame, is also non-inertial. Instead, if any frame is in translational accelerated motion relative to inertial frame, is accelerated or non-inertial frame.

Let us now discuss the two cases in detail

Reference frame with translational Acceleration:

Let us consider two non-inertial frames S and S' such that the frame S' is moving with acceleration $\vec{a}_0$ with respect to S. Let a Particle has an acceleration $\vec{a}$ with respect to S. Then to the observer in S', it will appear to have acceleration $\vec{a}'$ given by $\vec{a}' = \vec{a} - \vec{a}_0$

Hence the force, $\vec{F}' = m\vec{a}' = m(\vec{a} - \vec{a}_0)$
$= \vec{F} - m\vec{a}_0 = \vec{F} - \vec{F}_0$

where $\vec{F}$ is the force seen by an observer in S and $\vec{F}_0$ is the force due to relative acceleration $\vec{a}_0$ between two frames.

when $\vec{F} = 0$, we get $\vec{F}' = -\vec{F}_0$

Thus, the Particle seems to experience a force, $-\vec{F}_0$ when viewed from S' even when there is no force on it in S. Evidently an accelerated frame is a non-inertial frame & force $\vec{F}_0$ is called the fictitious or Pseudo force which arises from the acceleration of reference frame.

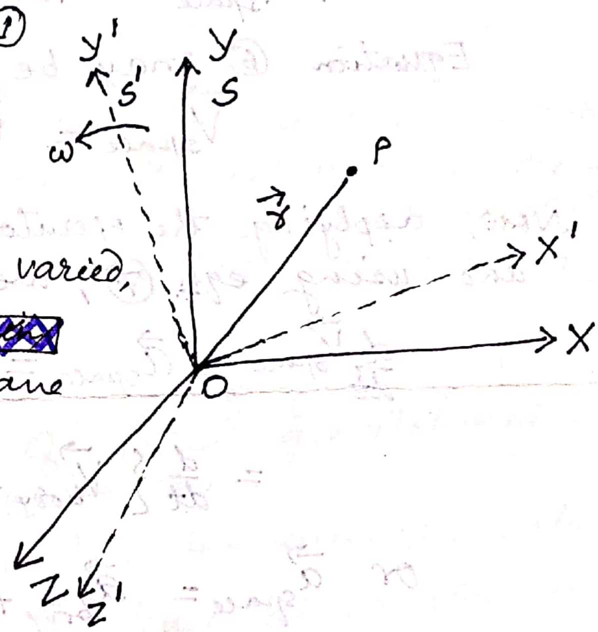
Uniformly Rotating Frame : Coriolis and Centrifugal force :

In this case ⓐ, two Conditions may arise -

One is - when the origins of two frames Coincide and the Other is - when the origins of two frames do not Coincide.

Let us Consider two frames S and S' in which S' is in uniform rotation w.r. to S at rest. Let the origins of the two frames Coincide at O. Also, let $\vec{r}$ be the radius vector of a Particle P moving in rotating frame.

then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ ---→ ①
$= x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$



where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors in S frame and $\vec{i}', \vec{j}'$ & $\vec{k}'$ are the unit vectors in S' frame.

In rotation unit vectors are varied,

So ~~[illegible]~~ we have

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x'\vec{i}' + y'\vec{j}' + z'\vec{k}')$$

$$= \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}'$$

$$+ x'\frac{d\vec{i}'}{dt} + y'\frac{d\vec{j}'}{dt} + z'\frac{d\vec{k}'}{dt} \quad \text{—— ②}$$

The first three terms in equation ② represent the velocity relative to S and remaining three terms gives the velocity of a Point attached to S'.

If $\vec{\omega}$ is the angular velocity of S' frame relative to S, then, according to $\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$, we get

$$\frac{d\vec{i}'}{dt} = \vec{\omega} \times \vec{i}', \quad \frac{d\vec{j}'}{dt} = \vec{\omega} \times \vec{j}' \text{ and } \frac{d\vec{k}'}{dt} = \vec{\omega} \times \vec{k}' \quad \text{—— ③}$$

Now Putting the values of eqn ③ in eqn (2), we obtain

$$\frac{d\vec{r}}{dt} = \frac{dx'}{dt}\vec{i}' + \frac{dy'}{dt}\vec{j}' + \frac{dz'}{dt}\vec{k}' + x'(\vec{\omega} \times \vec{i}') + y'(\vec{\omega} \times \vec{j}') + z'(\vec{\omega} \times \vec{k}') \quad \text{—— ④}$$

Equation ④ may be written as follows

$$\left(\frac{d\vec{r}}{dt}\right)_{space} = \left(\frac{d\vec{r}}{dt}\right)_{body} + \vec{\omega}\times\vec{r} \quad \text{———} \; ⑤$$

where $\left(\frac{d\vec{r}}{dt}\right)_{space}$ is the velocity of the rigid body with respect to S and $\left(\frac{d\vec{r}}{dt}\right)_{body}$ is the velocity with respect to S'. This result is actually true for any vector and can be represented by the following operator equation

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + (\vec{\omega}\times\;) \quad \text{———} \; ⑥$$

Equation ⑥ may be written as

$$\vec{V}_{space} = \vec{V}_{body} + \vec{\omega}\times\vec{r} \quad \text{———} \; ⑦$$

Now, applying the operator equation ⑥ to the vector $\vec{V}_{space}$ and using eqn ⑦, we get

$$\frac{d}{dt}\vec{V}_{space} = \vec{a}_{space} = \left(\frac{d\vec{V}_{space}}{dt}\right)_{body} + \vec{\omega}\times V_{space}$$

$$= \frac{d}{dt}\left\{\vec{V}_{body} + (\vec{\omega}\times\vec{r})_{body}\right\} + \vec{\omega}\times\left(\vec{V}_{body} + \vec{\omega}\times\vec{r}\right)$$

$$\text{or } \vec{a}_{space} = \vec{a}_{body} + 2(\vec{\omega}\times V_{body}) + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \frac{d\vec{\omega}}{dt}\times\vec{r} \quad \rightarrow ⑧$$

The acceleration in the rotating frame S' is

$$\vec{a}_{body} = \vec{a}_{space} - 2(\vec{\omega}\times\vec{V}_{body}) - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{d\vec{\omega}}{dt}\times\vec{r} \quad \text{———} \; ⑨$$

The equation of motion in the fixed space is

$$\vec{F}_{space} = m\vec{a}_{space}$$

$$\text{Hence } m\vec{a}_{body} = F_{space} - 2m(\vec{\omega}\times\vec{V}_{body}) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\frac{d\vec{\omega}}{dt}\times\vec{r} \quad \text{———} \; ⑩$$

obviously to an observer in the rotating frame, the body appears to be moving under the effective force $m\vec{a}_{body}$. The term $m\vec{\omega}\times(\vec{\omega}\times\vec{r})$ in eqn ⑩ is the ordinary centrifugal force and is perpendicular to $\vec{\omega}$. Its magnitude is $m\omega^2 r\sin\theta$

The term $2m(\vec{\omega} \times \vec{V}_{body})$ is called Coriolis force and is perpendicular to both $\vec{\omega}$ and $\vec{V}_{body}$. This is nonzero only when $\vec{V}_{body} \neq 0$ and the velocity of a point relative to the rotating frame must have a nonzero projection on a plane perpendicular to the axis of rotation. The last term $\left(\frac{d\vec{\omega}}{dt} \times \vec{r}\right)$ is nonzero only when $\frac{d\omega}{dt} \neq 0$ and will vanish when ω is constant.

It is now clear that the rotating frame is a non-inertial frame because the particle is acted upon by two fictitious forces as centrifugal and Coriolis force in addition to real force $\vec{F}_{space} = \vec{F}_{body}$. In case of inertial frame the motion has the simplest form $\vec{F} = m\vec{a}$.

Effect of Centrifugal force

The earth is rotating from west to east and a reference frame is fixed on it is a rotating frame with respect to a fixed star frame. Thus a particle at rest or in motion on the earth is acted upon by fictitious forces i.e. Centrifugal and Coriolis forces. Let us take first Centrifugal force —

The total apparent force of gravity acting on a pendulum is the sum of actual gravitational force and the centrifugal force as given by

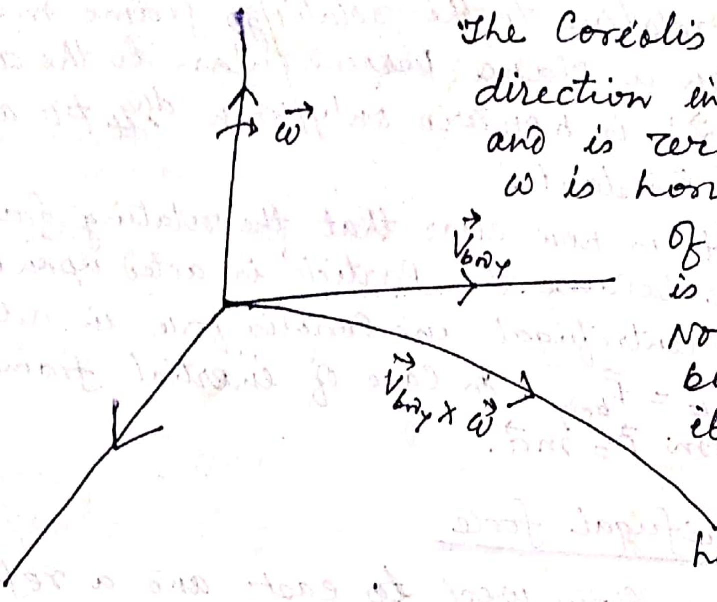
$$g' = g - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

. However the value of g will vary with latitude, being least at equator and greatest at the poles. A plumb line will not point exactly towards the centre of the earth but is swung through a small angle due to the centrifugal force. ~~Hence~~ However the actual measured value differs from theoretical value. This discrepancy is attributed to the fact that earth is not a perfect sphere, and is flattened at the poles. Thus the value of g itself is greater at poles than at the equator, the centrifugal term disregarded. The flattening of earth increases this tendency.

Effect of Coriolis force

The Coriolis force on a moving particle is perpendicular to both $\vec{\omega}$ and $\vec{V}_b$. In the Northern Hemisphere, where $\vec{\omega}$ point out of ground, the Coriolis force $2m(\vec{V}_{body} \times \vec{\omega})$

tends to deflect a Projectile shot along the earth's surface, to the right of its direction of travel.



The Coriolis deflection reverses direction in the Southern Hemisphere, and is zero at the equator where ω is horizontal. The magnitude of Coriolis acceleration ~~are~~ is always less than $15\ cm/sec^2$. Normally it is extremely low but there are instances where it becomes important.

The Coriolis force has deflecting action on the motion of air and water masses on the earth and thereby it affects the weather. The water of rivers in the northern hemisphere which flow along the direction of meridian ie from north to the South or vice-versa, experince a deflection towards the right bank with the Consequence that the right bank of such rivers is steeper than the left bank. The waters of a river that is flowing Southward have a velocity component perpendicular to the axis and directed away from it. If the river flows in a South to north direction, the deflection will be towards the east ie to the right again.

The warm Gulf stream which flows northwards is deflected towards the east, which has a great bearing on the climate of Europe.

Another Practical importance is the occurance of cyclones and tradewinds. whenever a regeon of low Pressure arises in the northern hemisphere, the air from the surrounding area gets sucked in owing to the Pressure gradient. As the air starts to move, the Coriolis force causes it to drift the right, causeing an anticlockwise rotation around the low Pressure zone. This Process continues till the thrust due to the Pressure gradient is balanced by that due to the Coriolis force. This Phenomenon causes cyclones.

It is also responsible for tradewinds. The heating of earth surface near equator causes the air to rise by convection currents and be replaced by cooler air flowing in towards the equator. The direction of flow is not North-South but due to Coriolis force it gets deviated towards the west. Thus we get the north-west trade winds in Northern hemisphere and similarly south-east trade wind in Southern hemisphere. Effects due to Coriolis force also appear in atomic Physics.

Gravitational Field and Potential

It is quite evident from Newton's law of gravitation that every body attracts another body. Naturally, the space, surrounding the body up to which force of attraction effectively works, is called the gravitational field.

The intensity of gravitational field of a body at a point is the force experienced by unit test mass placed at that point.

If a body of mass Δm be placed at a point in the gravitational field which experiences a force ΔF, the intensity of gravitational field is

$$E = \frac{\Delta F}{\Delta m}$$, if we put $\Delta m = 1$,

$$\vec{E} = \Delta \vec{F}.$$

If the ~~body~~ field be created by a body of mass M and ~~field~~ intensity at a point r distance apart, ~~then~~ will be

$$\boxed{E = \frac{GM}{r^2}}$$

Gravitational potential

The gravitational at a point is defined as the work done by the external agent in bringing unit test mass from infinity up to the point.

~~If a~~ b The gravitational potential, in the field of a body of mass M at a point r distance apart, will be.

$$\boxed{V = -GM/r}$$

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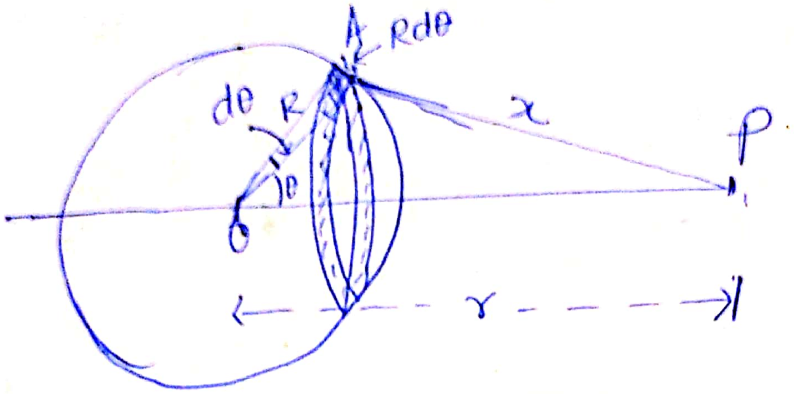
Evidently we see that the intensity of gravitational field E is related to Potential V as

$$\boxed{E = -\frac{dV}{dr}}$$

Potential and field at a point due to a Spherical Shell:



Let us Consider a spherical shell of Mass M and Radius R. P is a point at a distance r from the Centre O of the shell where Potential has to be Calculated.

For this, the shell is divided into Rings. Let us take one of them at right angles to OP making angle θ with Centre with thickness $R d\theta$.

The Surface area of the Ring

$$= (2\pi R \sin\theta)(R d\theta)$$

$$= 2\pi R^2 \sin\theta \, d\theta$$

The Mass of the Ring Considered

$$= 2\pi R^2 \sin\theta \, d\theta \times \frac{M}{4\pi R^2} \quad \text{where}$$

$$= \frac{M}{2} \sin\theta \, d\theta$$

$$\left[\frac{M}{4\pi R^2} = \text{Mass of the shell per unit Surface area}\right].$$

If x is the distance of P from a point of the ring, then according to the definition, the potential at P due to this ring is

$$dV = -\frac{G \cdot \frac{M}{2}\sin\theta\, d\theta}{x}$$

$$= -\frac{GM}{2x}\sin\theta\, d\theta.$$

From, the Cosine property of the triangle OAP,

$$x^2 = R^2 + r^2 - 2Rr\cos\theta$$

Differentiating it,

$$2x\,dx = 2Rr\sin\theta\, d\theta$$

$$\therefore \sin\theta\, d\theta = \frac{x\,dx}{Rr}$$

$\therefore$ The Potential at P is

$$dV = -\frac{GM}{2x}\sin\theta\, d\theta$$

$$= \frac{GM}{2Rr}\,dx \quad \text{—— (1)}$$

Now, the three Cases may arise.

I. When the Point P lies outside the shell.

The Potential at P due to the entire shell is obtained by integrating the eq. (1) from $x = R-r$ to $x = R+r$

$$\therefore V = \int_{R-r}^{R+r} dV = -\frac{GM}{2Rr}\int_{R-r}^{R+r} dx$$

$$= -\frac{GM}{2Rr}\left[x\right]_{R-r}^{R+r} = -\frac{GM}{\cancel{2}R\cancel{r}}\cdot\cancel{2r} = -\frac{GM}{r}$$

Thus $\boxed{V = -\frac{GM}{r}}$

Hence the field out-side the shell:

$$E = -\frac{dv}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2}$$

$$\therefore \boxed{E = -\frac{GM}{r^2}}$$

Case II. when P lies at the surface then $R = r$.

Naturally $\boxed{V = -\frac{GM}{R}}$

and field $\boxed{E = GM/R^2}$.

III. when P lies inside the shell.

In this case the potential is found by integrating eqn. (1) from $x = r - R$ to $r + R$

$$\therefore V = \int_{r-R}^{r+R} dv = -\frac{GM}{2Rr}\int_{r-R}^{r+R} dx$$

$$= -\frac{GM}{2Rr}\left[x\right]_{r-R}^{r+R}$$

$$= \frac{-GM}{\cancel{2}R\cancel{r}} \cancel{2r} = -\frac{GM}{R}$$

$$\therefore \boxed{V = -GM/R}$$

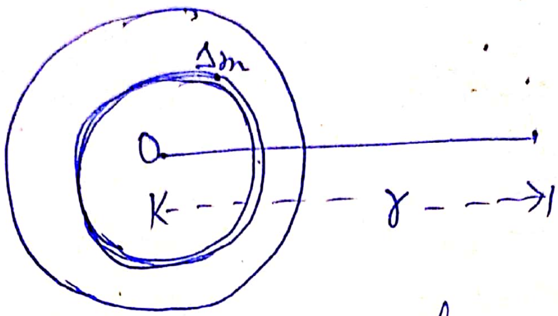
This expression clearly shows that the potential remains the same as it was at the surface.

Naturally, the field inside the shell

$$E = -\frac{dv}{dr} = -\frac{d}{dr}\left(-\frac{GM}{R}\right)$$

$$= \text{Zero}.$$

Gravitational Potential and field due to a solid sphere



Let us Consider a homogeneous Solid sphere of mass M and Radius R. P is a point at a distance r from the Centre O of the sphere.

Case I — When the point P lies outside the sphere:

The solid sphere may be supposed to be divided into thin spherical shells of equal mass Δm. Since P lies outside this spherical shell, ~~it~~ the potential at P due to the shell

$$= -\frac{G\Delta m}{r}.$$

$\therefore$ The potential at P due to the entire sphere

$$V = -\sum \frac{G\Delta m}{r} = -\frac{G}{r}\sum \Delta m$$

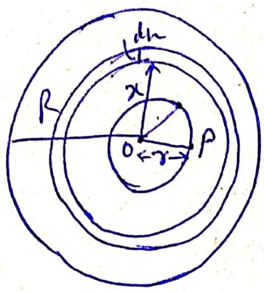
$$= -\frac{GM}{r}.$$

Thus, the gravitational field

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right)$$

$$= -GM/r^2$$

Case II. When the point P lies inside the sphere.



Let us imagine a concentric spherical surface through P. The potential at P arises out of inner sphere and outer thick spherical shell.

Let $V = V_1 + V_2$, where V_1 = potential due to inner sphere of radius r and V_2 = potential due to outer thick shell.

The mass of inner sphere $= \frac{4}{3}\pi r^3 \rho$ where ρ = density of the sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}.$$

The potential at P due to this sphere

$$V_1 = -\frac{G\left(\frac{4}{3}\pi r^3 \rho\right)}{r} = -\frac{4\pi}{3} G \rho r^2 \quad (1)$$

To evaluate V_2, let us consider a thin concentric shell of radius x and thickness dx.

The volume of the shell $= 4\pi x^2 dx$

and its mass $= 4\pi x^2 dx \rho$

The potential at P due to the shell

= self potential of the shell

$$= -\frac{G(4\pi x^2 dx \rho)}{x}$$

$$= -G\, 4\pi \rho x\, dx.$$

$\therefore$ the Potential at P due to the outer thick shell.

$$= -\int_{x=r}^{x=R} 4\pi G \rho x\, dx$$

$$= -4\pi G\rho \left[\frac{x^2}{2}\right]_r^R = -G\, 4\pi \rho \left[\frac{R^2-r^2}{2}\right]$$

$$= 2\pi G\rho (R^2 - r^2) \quad \text{—— (II)}$$

$$\therefore V = V_1 + V_2 = -\frac{4\pi}{3} G\rho r^2 - 2\pi G\rho (R^2 - r^2)$$

$$= -\frac{4\pi}{3} G\rho \left[r^2 + \frac{3}{2}R^2 - \frac{3}{2}r^2\right]$$

$$= -\frac{\cancel{4}}{\cancel{3}}\cancel{\pi} G \cdot \frac{\cancel{3}M}{\cancel{4}\cancel{\pi}R^3}\left(\frac{3R^2 - r^2}{2}\right)$$

$$= -\frac{GM}{2R^3}(3R^2 - r^2)$$

Therefore, the gravitational field inside the sphere.

$$E = -\frac{dv}{dr} = -\frac{d}{dr}\left\{-\frac{GM}{2R^3}(3R^2 - r^2)\right\}$$

$$= -\frac{GM}{2R^3} \cdot 2r = -\frac{GM}{R^3} \cdot r$$

$$\boxed{E = -\frac{GM}{R^3} \cdot r}$$

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