Principle of least Action

Another variational Principle associated with the Hamiltonian formulation is known as the Principle of least Action. In mechanics, action is a quantity defined most generally as

$$A = \int_{t_1}^{t_2} 2T\,dt = \int_{t_1}^{t_2} \sum_k P_k \dot{q}_k\,dt \quad \text{---} \quad (1)$$

Therefore the Principle of Least Action states that in a System for which H is Conserved

$$\Delta \int_{t_1}^{t_2} \sum_k P_k \dot{q}_k\,dt = 0 \quad \text{-----} \quad (2)$$

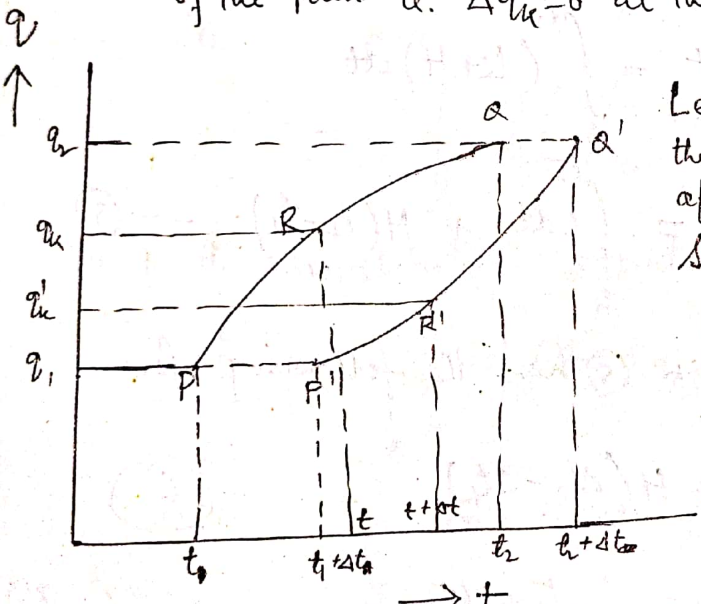
where Δ represents a new type of variation of Path

In order to deduce this Principle, we have to use Δ variation in which

(I) Time as well as position Co-ordinates are allowed to vary

(II) Time varies even at end points of the Path

(III) The position Co-ordinates are held fixed at the end points of the Path i.e. $\Delta q_k = 0$ at the end points.



Let PRQ be the actual Path and P'R'Q', the varied Path. The end points P & Q after time δt takes the Position P' & Q' such that position Co-ordinates of P and Q are fixed while the time t is not fixed. A point R on actual Path now goes on R' with the Correspondence

$$q_k \rightarrow q'_k = q_k + \Delta q_k.$$

If α is the variational Parameter, then in δ process t is independent of α but in Δ process t is ~~even~~ function of α even at end points i.e. $t = t(\alpha)$

Thus the function q_k depends upon t and α throughout

i.e. $q_k = q_k(t, \alpha)$

Analytically Δ variation is defined as

$$\Delta q_k = \left[\frac{d}{d\alpha} q_k(\alpha, t)\right] d\alpha = \left[\frac{\partial q_k}{\partial \alpha} + \frac{dq_k}{dt}\frac{dt}{d\alpha}\right] d\alpha$$

$$= \frac{\partial q_k}{\partial \alpha} d\alpha + \dot{q}_k \frac{dt}{d\alpha} d\alpha$$

But $\delta q_k = \frac{\partial q_k}{\partial \alpha} d\alpha$ and $\dot{q}_k \frac{dt}{d\alpha} d\alpha = \dot{q}_k \Delta t$

So $\Delta q_k = \delta q_k + \dot{q}_k \Delta t$ —— (3)

The relation beetween Δ-variation and δ-variation can now shown to hold for any function $f(q_k, t)$ as

$$\Delta f = \delta f + \dot{f} \Delta t \quad \left[\Delta f = \sum_k \frac{\partial f}{\partial q_k} \Delta q_k + \frac{\partial f}{\partial t} \Delta t \right.$$

Thus. $\Delta = \delta + \Delta t \frac{d}{dt}$ —— (4) $\quad = \sum_k \frac{\partial f}{\partial q_k} (\delta q_k + \dot{q}_k \Delta t) + \frac{\partial f}{\partial t} \Delta t$

It is to be noted that the $\quad \left. = \delta f + \dot{f} \Delta t \right]$

Δ operation and time differentiation can't be interchanged in this case which is done in δ-variation

Now from eqn. (1), we have

$$A = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt = \int_{t_1}^{t_2} (L + H) \, dt$$

$$= \int_{t_1}^{t_2} L dt + H(t_2 - t_1) \quad \text{—— (5)}$$

The Δ-variation of eqn. (5) has the following form

$$\Delta A = \Delta \int_{t_1}^{t_2} L dt + H(\Delta t_2 - \Delta t_1) \quad \text{—— (6)}$$

Let us now solve the integral

$$\Delta \int_{t_1}^{t_2} L dt$$

It is also rembembured that t_1 & t_2 limits are also subjected to change in this variation, Δ can't be taken inside the integral.

let $\int_{t_1}^{t_2} L dt = I$

So

$$\Delta I = \delta I + \dot{I}\Delta t \quad \text{from eqn } (4)$$

Therefore $\Delta \int_{t_1}^{t_2} L\,dt = \Delta I(t_2) - \Delta I(t_1)$

$$= \left[\delta I(t_2) + \dot{I}(t_2)\Delta t_2\right] - \left[\delta I(t_1) + \dot{I}(t_1)\Delta t_1\right]$$

$$= \delta I(t_2) - \delta I(t_1) + \dot{I}(t_2)\Delta t_2 - \dot{I}(t_1)\Delta t_1$$

$$= \delta \int_{t_1}^{t_2} L\,dt + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——} \quad (7)$$

From the nature of δ- variation, we have

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\left(\sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right) dt$$

which by Lagrange's equations can be written

$$\delta \int_{t_1}^{t_2} L\,dt = \sum_k \int_{t_1}^{t_2} \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\delta q_k)\right\} dt$$

$$= \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt$$

with substitution equation (7) becomes

$$\Delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——} \quad (8)$$

using eqn. (3) to the first integral term of eqn (8)

we have

$$\Delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_k \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\Delta t\right) dt \quad \left[\delta q_k = \Delta q_k - \dot{q}_k\Delta t\right]\right.$$

$$+ \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\sum_k \frac{\partial L}{\partial \dot{q}_k}\Delta q_k - \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2} \quad \text{——} \quad (9)$$

At the end points $\Delta q_k = 0$ but Δt does not. So equation (9) becomes

$$\Delta\int_{t_1}^{t_2} L\,dt = \left[-\sum_k \frac{\partial L}{\partial \dot q_k}\dot q_k \Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-\sum_k p_k \dot q_k \Delta t\right]_{t_1}^{t_2} + \left[L\Delta t\right]_{t_1}^{t_2}$$

$$= \left[\left(L - \sum_k p_k \dot q_k\right)\Delta t\right]_{t_1}^{t_2}$$

$$= \left[-H\Delta t\right]_{t_1}^{t_2} \qquad (10)$$

If we restrict to the system for which H is const. then $\frac{\partial H}{\partial t} = 0$

Thus $\left[H\Delta t\right]_{t_1}^{t_2} = \Delta\int_{t_1}^{t_2} H\,dt$

Substituting this in eq. (10), we get $\Delta\int_{t_1}^{t_2} L\,dt = -\Delta\int_{t_1}^{t_2} H\,dt$ *

Combining these terms, the total variation of action is

$$\Delta A = \left[\left(-\sum_k p_k \dot q_k + L + H\right)\Delta t\right]_{t_1}^{t_2}$$

$$= \left[(-H + H)\Delta t\right]_{t_1}^{t_2} = 0 \qquad (11)$$

This completes the proof of principle of least action

* $\Delta\int_{t_1}^{t_2}(L+H)\,dt = 0$

or $\Delta\int_{t_1}^{t_2}\left(\cancel{L} + \sum_k p_k \dot q_k - \cancel{L}\right)dt = 0$

$= \Delta\int_{t_1}^{t_2}\sum p_k \dot q_k\,dt = 0 \Rightarrow \Delta\int_{t_1}^{t_2} 2T\,dt = 0$

which is the principle of least action.

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Variational or Hamilton's Principle

Derive Lagrangian Equation of Motion from this Principle

A more general formulation of the Lagrangean in mechanics is due to the variational principle (so called Hamilton's principle). The Principle is stated in a generalised form independent of any Co-ordinates system and hence is useful in non-mechanical systems and fields also. It is also known as integral principle.

This principle may be stated as — "Out of all the possible paths along which a dynamical system may move from one point to another point within a given interval of time (Consistent with Constraints if any), the actual path followed is that which minimizes the time integral of the Lagrangian."

Analytically it can be represented as

$$I = \int_{t_1}^{t_2} L\,dt = \text{extremum} \quad \text{—— (1)}$$

where I is the extremum value of time integral of Lagrangean and is known as Hamilton's principal function of the Path.

Taking δ-variation of the eqn. (1), the variational principle may also be represented as

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{—— (2)}$$

where $L = T - V$, $T = K.E$ & $V = P.E.$

In order to deduce Lagrangean Eqn. of Motion from this Principle, let us take a Conservative System of particles for which Lagrangian is given by

$$L = L(q_k, \dot{q}_k, t) \quad \text{—— (3)}$$

But due to homogeneity of time, the Lagrangian for a dynamical system is not an explicit fn of time. Thus, the eqn (3) reduces to

$$L = L(q_k, \dot{q}_k) \quad \text{—— (4)}$$

~~Now using~~ Hence $\delta L = \sum_k \frac{\partial L}{\partial q_k}\delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k$ —— (5)

Now using eqn. ⑤ in eqn. ②, we obtain

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\delta q_k)\, dt = 0 \quad \text{Since } \delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$$

or $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \sum_k \left[\left\{ \frac{\partial L}{\partial \dot{q}_k} \delta q_k \right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k\, dt \right] = 0$$

(Integrating by Parts —— ⑥

But $\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$ because $\delta q_k = 0$ at end points

Therefore eqn. ⑥ reduces to

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt - \sum_k \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k\, dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right] \delta q_k\, dt = 0 \qquad \text{—— ⑦}$$

But variables being independent, the variations δq_k independent if and only if

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

i.e. $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

Which is Langrange's equation of Motion

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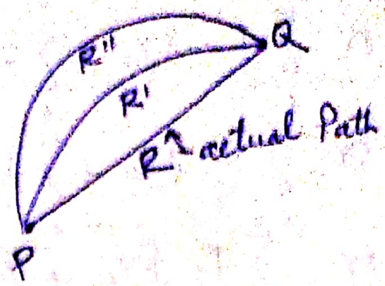
Deduction of Hamilton's Principle

Let us consider that the conservative holonomic dynamical system moves from P to Q where P & Q are initial and final points at time t_1 & t_2 respectively. Let PRQ be the actual path and PR'Q & PR''Q be the two neighbouring paths out of infinite no. of possibilities.



For the deductions of this principle there must be satisfied two following conditions —

δt must be zero at end points

and δr must be zero at end points.

Let the ith particle of the system be acted upon by a no. of forces given by $\vec{F_i}$ acquiring accleration $\ddot{\underline{r}}_i$, so that we have

$$\vec{F_i} = m_i \ddot{\underline{r}}_i$$

From D' Alembert's Principle, we have

$$\sum_i (\vec{F_i} - m_i \ddot{\underline{r}}_i)\delta \underline{r}_i = 0$$

$$\text{or} \quad \sum_i \vec{F_i} \cdot \delta \underline{r}_i - \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = 0 \quad \text{—— (1)}$$

$$\text{But} \quad \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \frac{d}{dt}(\delta \underline{r}_i) \quad \text{—— (2)}$$

If there is a little variation along the actual and neighbouring paths, we have

$$\frac{d}{dt}(\delta \underline{r}_i) = \delta \frac{d}{dt}(\underline{r}_i) = \delta \dot{\underline{r}}_i \quad \text{—— (3)}$$

using eqn (3), eqn (2) may be written as

$$\ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \quad \text{—— (4)}$$

using eqn (4), eqn (1) becomes

$$\sum_i \underline{F}_i \cdot \delta \underline{r}_i - \sum_i m_i \left[\frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \right] = 0$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i - \sum_i m_i \left[\frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \frac{1}{2}\delta(\dot{\underline{r}}_i^2) \right] = 0$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i + \sum_i \frac{1}{2} m_i \delta(\dot{\underline{r}}_i^2) = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

or $$\sum_i \underline{F}_i \cdot \delta \underline{r}_i + \delta \sum_i \left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right) = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i) \quad (5)$$

But $\sum_i \underline{F}_i \cdot \delta \underline{r}_i = \delta W$ = work done by the forces $\underline{F}_i$ during displacement $\delta \underline{r}_i$

and $\delta \sum_i \left(\frac{1}{2} m_i \dot{\underline{r}}_i^2\right) = \delta T$ where T is K.E

Therefore eqn. (5) becomes

$$\delta W + \delta T = \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

Integrating above eqn between limits t_1 & t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)\, dt$$

$$= \sum_i \int_{t_1}^{t_2} d(m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)$$

$$= \sum_i (m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i)_P^Q = 0$$

Since $\delta \underline{r}_i$ at P & Q is zero

For a Conservative System, we know that

$\delta W = -\delta V$ where V is Potential Energy

$$\therefore \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} \delta(T - V)\, dt = 0$$

$$\text{or} \quad \delta \int_{t_1}^{t_2} L\, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} L\, dt = \text{extremum}$$

which is Hamilton's Principle

Hamilton's Equations of motion from Hamilton's Principle (variational Principle)

We have already introduced Hamiltonian function expressed as

$$H = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \quad \text{—— (3)}$$

or

$$L = \sum_k p_k \dot{q}_k - H \quad \text{—— (4)}$$

so

$$\delta L = \delta \sum_k p_k \dot{q}_k - \delta H$$

or

$$\delta L = \sum_k \delta p_k \dot{q}_k + \sum_k p_k \delta \dot{q}_k - \delta H \quad \text{—— (5)}$$

Now putting the value of δL from eqn (5) in eqn. (2) of Variational Principle, we get

$$\int_{t_1}^{t_2} \left(\sum_k \delta p_k \dot{q}_k + \sum_k p_k \delta \dot{q}_k - \delta H \right) dt = 0 \quad \text{—— (6)}$$

Since Hamiltonian is not an explicit fn of time, then

$H = H(q_k, p_k, t)$ can be written as $H = H(q_k, p_k)$ —— (7)

Differentiating eqn (7) Partially, we get

$$\delta H = \sum_k \left(\frac{\partial H}{\partial q_k} \delta q_k + \frac{\partial H}{\partial p_k} \delta p_k \right) \quad \text{—— (8)}$$

Now putting the values of δH from eqn. (8) in eqn. (6), we get

$$\int_{t_1}^{t_2} \sum_k \left(\delta p_k \dot{q}_k + p_k \delta \dot{q}_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt = 0 \quad \text{—— (9)}$$

Integrating 2nd term of eqn. (9) by Parts,

$$\int_{t_1}^{t_2} p_k \delta \dot{q}_k \, dt = \int_{t_1}^{t_2} p_k \frac{d}{dt} (\delta q_k) \, dt$$

$$= \left| p_k \delta q_k \right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt$$

$$= 0 - \int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt$$

First term is zero because at end points $\delta q_k = 0$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}(p_k)\, \delta q_k \, dt \quad \text{--- (10)}$$

Now using eqn (10) in eqn (9), we get

$$\int_{t_1}^{t_2} \sum_k \left(\delta p_k \dot{q}_k - \frac{d}{dt}(p_k)\, \delta q_k - \frac{\partial H}{\partial q_k} \delta q_k - \frac{\partial H}{\partial p_k} \delta p_k \right) dt = 0$$

or $$\int_{t_1}^{t_2} \sum_k \left\{ \left(\dot{q}_k - \frac{\partial H}{\partial p_k} \right) \delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k} \right) \delta q_k \right\} dt = 0$$

In order to satisfy the above equation, each integrand must vanish as δp_k and δq_k are arbitrary variations

i.e. $$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0$$

or $$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{--- (11)}$$

and $$\dot{p}_k + \frac{\partial H}{\partial q_k} = 0$$

or $$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{--- (12)}$$

Equations (11) and (12) are called Hamilton's equations of motion.