Vector Space over a field V(F):- Let V be a non-empty set and F a field. Then these two composition as follow.

(i) Binary composition in V by +, so that $u+v \in V$, whenever $u \in V$ & $v \in V$.

(ii) Scalar multiplication $\alpha u \in V$, for each $\alpha \in F$ and $u \in V$.

Then following these properties when (V,+) is abelian group.

(i) = $\alpha(u+v) = \alpha u + \alpha v$

(ii) $(\alpha+\beta)u = \alpha u + \beta u$

(iii) $\alpha(\beta u) = (\alpha\beta)u$

(iv) $1u = u \quad \forall u \in V$

Subspace of a vector space:- Let V be a vector space over a field F. A non-empty subset W of V is called a subspace of V, if W under the compositions of V, is a vector space over F.

W is a subspace of V whenever $w_1, w_2 \in W$ and $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$

Theorum A non-empty subset W of a vector space V(F) is a subspace ~~of~~ V if and only if

(i) $w_1 \in W$ and $w_2 \in W \Rightarrow w_1 + w_2 \in W$ & $w_1 - w_2 \in W$

(ii) $\alpha \in F$ and $w \in W \Rightarrow \alpha w \in W$

Theorum- A non-empty subset W of a vector space V(F) is a subspace of V if and only if $\alpha w_1 + \beta w_2 \in W \ \forall \alpha, \beta \in F$ and $w_1, w_2 \in W$

Theorem- The intersection (∩) of two subspaces of a Vector space V(F) is a subspace of V(F)

Remark- The union of two subspaces of a vector space V(F) need not be a subspace of V(F).

Linear Combination: If V is a vector space over F and if $v_1, v_2, v_3, \dots v_n \in V$ then an expression of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, is called a linear combination of the vectors $v_1, v_2, v_3, \dots v_n$ over F, where $\alpha_i \in F$ and $v_i \in V$.

Linear Span $L(S)$: If S is a non-empty subset of the vector space $V(F)$, then the linear span of S, $L(S)$ is the set of all linear combination of finite sets of elements of S.

$$L(S) = \left\{ \alpha_1 s_1 + \alpha_2 s_2 + \dots + \alpha_n s_n : \alpha_i \in F,\ s_i \in S,\ 1 \le i \le n,\ n \in N \right\}$$

Exp (i) Let $S = \{(1,0), (0,1)\}$ be a subset of $R^2(R)$. Show that $(3,5)$ belongs to $L(S)$

(ii) Let $S = \{(1,0,0), (0,1,0)\} \subseteq R^3(R)$ $\xrightarrow{3}$ 3-tuples/dimension/elements of set; $(R) \rightarrow$ numbers

Find $L(S)$. Do $(3,2,0)$ and $(3,5,1)$ belongs to $L(S)$?

Solution

We observe that

(i) $(3,5) = 3(1,0) + 5(0,1)$ and so $(3,5) \in L(S)$

(ii) $L(S) = \{\alpha(1,0,0) + \beta(0,1,0) : \alpha, \beta \in R\}$

$= \{(\alpha+0, 0+\beta, 0+0) : \alpha, \beta \in R$

$= \{(\alpha, \beta, 0) : \alpha, \beta \in R\}$

It is obvious that $(3,2,0) \in L(S)$ and $(3,5,1) \notin L(S)$
$(\because 1 \neq 0)$

(iii) Let $S = \{(2,3), (1,4)\} \subseteq R^2(R)$. Show that $(4,1) \in L(S)$.

Let $(4,1) = \alpha(2,3) + \beta(1,4) : \alpha, \beta \in R$

$= (2\alpha + \beta, 3\alpha + 4\beta)$

$2\alpha + \beta = 4$ and $3\alpha + 4\beta = 1$

$\alpha = 3$ and $\beta = -2$ satisfied the above equation

Finite-dimensional vector space over F.

The vector space V is said to be finite dimensional over F if there is a finite subset S in V such that $V = L(S)$

$F^{(n)}$ is finite-dimensional over F, for if S consists of the n vectors $(1,0,0,\cdots,0), (0,1,0,\cdots,0)\cdots,(0,0,0,\cdots$ then $V = L(S)$

Thus. $L(S)$ is the set of Linear Combinations of finite number of elements of S.

Remark - $S \subseteq L(S)$

Subspace Spanned by a Set :- If S is a non-empty subset of $V(F)$ such that $L(S) = W$, we say that the subspace W is spanned by S or S spanns W

Exp. Is the vector $(3,-4,6)$ in the subspace of R^3 spanned by the vectors $(1,2,-1), (2,2,1)$ and $(1,-2,3)$?

Solution Let $W = L(S)$ be the subspace of R^3 spanned by

$$S = \{ V_1 = (1,2,-1), V_2 = (2,2,1), V_3 = (1,-2,3) \}$$

Let $V = (3,-4,6) = \alpha V_1 + \beta V_2 + \gamma V_3$ where $\alpha, \beta, \gamma \in R$

Then

$$(3,-4,6) = \alpha(1,2,-1) + \beta(2,2,1) + \gamma(1,-2,3)$$
$$= (\alpha+2\beta+\gamma,\ 2\alpha+2\beta-2\gamma,\ -\alpha+\beta+3\gamma) \text{ --- (I)}$$
$$\alpha + 2\beta + \gamma = 3 \text{ --- (II)}$$
$$2\alpha + 2\beta - 2\gamma = -4 \text{ --- (III)}$$
$$-\alpha + \beta + 3\gamma = 6 \text{ --- (IV)}$$

Solving the Eqn (II), (III) & (IV) $\alpha = 2,\ \beta = -1$ & $\gamma = 3$

Substituting the value of α, β & γ in Eqn ①

$(3,-4,6) = 2(1,2,-1) + (-1)(2,2,1) + 3(1,-2,3)$

$= (2,4,-2) + (-2,-2,-1) + (3,-6,9)$

$= (2-2+3, 4-2-6, -2-1+9)$

$= (3,-4,6)$

Hence $V \in W = L(S)$

Exp. Show that the vector $(2,-5,3)$ is not in the subspace of R^3 spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$ and $(1,-5,7)$.

Solution Let $V = (2,-5,3) \in W$

$S = \{V_1 = (1,-3,2), V_2 = (2,-4,-1), V_3 = (1,-5,7)\}$

Let $V = \alpha V_1 + \beta V_2 + \gamma V_3$

$(2,-5,3) = \alpha(1,-3,2) + \beta(2,-4,-1) + \gamma(1,-5,7)$ — ①

$= (\alpha, -3\alpha, 2\alpha) + (2\beta, -4\beta, -\beta) + (\gamma, -5\gamma, 7\gamma)$

$= (\alpha + 2\beta + \gamma, -3\alpha - 4\beta - 5\gamma, 2\alpha - \beta + 7\gamma)$

$\alpha + 2\beta + \gamma = 2$ — (ii)

$-3\alpha + 4\beta - 5\gamma = -5$ — (iii)

$2\alpha - \beta + 7\gamma = 3$ — (iv)

From $3 \times$ (ii) + (iii), $2\beta - 2\gamma = 1 \Rightarrow \beta - \gamma = 1/2$ — ⑤

From $2 \times$ (ii) − (iv), $5\beta - 5\gamma = 1 \Rightarrow \beta - \gamma = 1/5$ — ⑥

Eqn ④ & ⑤ are inconsistance, so we cannot find values of α, β, γ clearly.

Hence the vector $(2,-5,3)$ cannot be expressed as a linear combination of the vectors $(1,-3,2)$, $(2,-4,-1)$ and $(1,-5,7)$.

Hence vector $(2,-5,3)$ is not in the subspace spanned by vectors.