Periodic flow of heat

When one end of a long metal bar is heated and cooled alternatively then it is called periodic flow of heat . We consider that the temperature varies simple harmonic manner .This method is applied to determine the thermal conductivity of a metal bar .

Fourier equation for one dimensional flow of heat is applied where radiation loss is neglected .

$$\frac{d\theta}{dt} = h\, d^2\theta / dx^2 \qquad (1)$$

Where Diffusivity , h = $\frac{K}{\rho s}$

In order to solve this equation we get

$$\theta = u + v \quad (2)$$

Wher u and v satisfy the relation

$$d^2 u / dx^2 = 0 \quad (3)$$

and

$$h\, d^2 v / dx^2 = dv / dt \quad (4)$$

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These equation shows that u is a function and v is a function of x and t both .

The solution of (3) gives the steady temperature at any point on the bar . Thus

$$u = \theta_1 - (\theta_1 - \theta_2 / l)\, x$$

where θ_1 and θ_2 are mean temperature at the ends of the bar of length l .

Let us now solve (4) ,We have the conditions that

$$v = v_0\, e^{\omega t} \qquad (5)$$

To solve (4) we can assume a solution of the form

$$\Psi = A\, e^{i\alpha t+\beta x}$$

Where A , α and β are constants , then

$$\frac{dv}{dt} = i\, \alpha\, A\, e^{i\alpha\alpha+\beta x}$$

$$= i\, \alpha\, v$$

$$d^2 v / dx^2 = \beta\, A\, e^{i\alpha\alpha+\beta x}$$

$$= \beta^2\, v$$

Substituting these values in (4) , we have

$$i\,\alpha = h\,\beta^2$$

$$\beta = \pm \sqrt{i\alpha/h}$$

Where $i = \sqrt{-1}$, which can be written as

$$I = ½\,(1 + i)^2$$

Then , $\beta = \pm \; (\alpha /2h)^{1/2}\,(1+ i)$

$$A = (\alpha /2h)^{1/2}$$

and we have

$$v = A\, e^{\pm a(1+i)x + i\alpha t)}$$

The positive sign a physically not allowed , as v = ∞ when x = ∞ . we take only the negative sign . Thus we get

$$v = Ae^{-ax} \; e^{i(\alpha t - ax)} \qquad (6)$$

At x = 0 , this equation reduces to

$$v = A\, e^{i\alpha t} \qquad (7)$$

But we know that at x = 0

$$v = v_0\, e^{i\alpha t} \qquad (8)$$

These two are equivalent , giving

$A = v_0$

$\alpha = \omega$

Therefore v can be written as

$$v = v_0 \, e^{-ax} \, e^{i(\omega t - ax)} \qquad (9)$$

considering only the real part , it can be expressed as

$$v = v_0 \, e^{-ax} \cos(\omega t - ax) \qquad (10)$$

In order to make this equation more general , we introduce an arbitrary constant δ in the cosine term . Then we have

$$v = v_0 \, e^{-ax} \cos(\omega t - ax + \delta) \qquad (11)$$

This represents a damped progressive wave . A damped progressive wave having the wave length λ and time period T along x axis is given by

$$v = v_0 \; e^{-ax} \cos [2\pi (t/T - x/\lambda + \delta)] \qquad (12)$$

comparing (11) and (12) , we get

$\omega = 2\pi/ T$

$a = 2\pi/\lambda$

and the velocity of the wave

$c = \lambda / T = \omega / a$

where ω is determined from the time period of heating and a is given by