Linear Vector Space

An arbitrary manifold R as set of vectors a, b, c, ... is defined as linear vector space endowed with metric if the following conditions are satisfied :-

(I) $a \in R \Rightarrow \lambda a \in R$ for any complex no. λ (closer property)

(II) $(\lambda_1 + \lambda_2) a = \lambda_1 a + \lambda_2 a$

(III) $\lambda_1 \lambda_2 a = \lambda_1 (\lambda_2 a)$

(IV) $a, b \in R \Rightarrow (a+b) = (b+a) = c \in R$

(V) $\lambda (a+b) = \lambda a + \lambda b$

(VI) $a + (b+c) = (a+b) + c$ (Associative)

(VII) $a + b = 0 \Rightarrow b = -a$ (Uniqueness of zero element)

(VIII) Inner product of any pair of vectors
$$(a, b) = (b, a)^*$$

(IX) $(a, b+c) = (a, b) + (a+c)$

(X) $(a, \lambda b) = \lambda (a, b) \Rightarrow (\lambda, a, b) = \lambda^* (a, b)$.

* The scalar product of any element of linear space (i.e vector) with itself is the norm of the vector. $|a|^2 = (a, a)$, which is a real no. in view of condition (VII).

* The +ve sq. root of $|a|^2 = (a, b)$ is the length of vector a, if $(a, a) = 1$, then the vector a is said to be normalised to unity.

* The vectors a and b are orthogonal to each other if
$$(a, b) = 0 = (b, a)$$

* For the vector space defined here with the condition $|a|^2 \geq 0$, one may readily prove that

$$|a| \cdot |b| \geq |(a, b)|$$

which is Schwartz inequality

* A set of n vectors $\vec{\alpha}_i$ $(i = 1, \ldots n)$ is said to be linearly independent if

$$\sum_{i=1}^{n} \lambda_i \alpha_i = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots \lambda_n = 0.$$

* The maxm. no. of such linearly independent vectors possibles in a linear space gives the dimensionality of the space. It may be finite or infinite.

In a finite linear space, a maxm. set of n linearly independent vectors is called a complete set. For such spaces with the condition $|a|^2 > 0$, Schmidt showed that one can find appropriate linear combinations of a linearly independent vectors and obtain another linearly independent complete set such that

$$(\vec{\alpha}_i, \vec{\alpha}_j) = \delta_{ij}, \quad i, j = 1 \ldots n.$$

i.e, orthonormal set of linearly independent vectors. Such a set is called set of basis vectors of the n-dimensional linear space.

For finite space the orthonormality and completeness of vectors are equivalent.

On the other hand, in infinite dimensional linear space completeness allows orthonormalization but completness does not follow from orthonormality in general.