Q2) Show that a necessary and sufficient condition for the coplanarity of three non-zero vectors is that their scalar triple product is zero.

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors. Then we know that the scalar triple product $\vec{a} \cdot \vec{b} \times \vec{c}$ = volume of the parallelopiped formed with $\vec{a}, \vec{b}, \vec{c}$ as coterminus edges. ——— Q.(1)

Necessity :-

Let $\vec{a}, \vec{b}, \vec{c}$ be coplanar. Then the height of the parallelopiped ODB is zero and hence the volume of the parallelopiped is zero.

$\therefore$ From (1), the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is zero.

Sufficiency :-

Let the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ be zero. Then the volume of the parallelopiped is zero.

$\therefore$ From (1) the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is zero.

Sufficiency :- Let the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ be zero. Then the volume of the parallelopiped is zero. Hence $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Another (Independent) Proof :-

Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero vectors.

Necessity :- Suppose $\vec{a}, \vec{b}, \vec{c}$ are coplanar. Let their plane be π. Now $\vec{b} \times \vec{c}$ is perpendicular to π. Since $\vec{a}$ lies in π. This means that $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$.

$$\therefore \vec{a} (\vec{b} \times \vec{c}) = 0$$

i.e. the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is zero.

Sufficiency :- Let $\vec{a} \cdot \vec{b} \times \vec{c} = 0$

Then $\vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Hence $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$.

$$\therefore \vec{a}, \vec{b}, \vec{c} \text{ are Coplanar.}$$