Exp. To find nth derivative (differential coefficient) of $e^{ax}\sin(bx+c)$.

Solution Let $y = \underset{I}{e^{ax}}\,\underset{II}{\sin(bx+c)}$ — ①

on differentiating w.r.to x Eqn ①

Then $\frac{dy}{dx} = y_1 = e^{ax}\cdot\frac{d}{dx}[\sin(bx+c)] + \sin(bx+c)\frac{d}{dx}e^{ax}$

$y_1 = e^{ax}\cdot\cos(bx+c).b + \sin(bx+c).e^{ax}.a$

$y_1 = e^{ax}[b\cos(bx+c) + a\sin(bx+c)]$

Let $a = r\cos\theta$ & $b = r\sin\theta$, so that

$a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2.1 = r^2$

$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \Rightarrow \tan\theta = \frac{b}{a} \quad \theta = \tan^{-1}\frac{b}{a}$

$\because y_1 = e^{ax}[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)]$

$= e^{ax}\cdot r[\cos\theta.\sin(bx+c) + \sin\theta\cos(bx+c)]$

$= re^{ax}[\frac{1}{2}.2\cos\theta.\sin(bx+c) + \frac{1}{2}.2\sin\theta\cos(bx+c)]$

$= re^{ax}.\frac{1}{2}[\sin(\theta+bx+c) - \sin(\theta - bx - c) + \sin(\theta+bx+c) + \sin(\theta - \cancel{bx} - c)]$

$\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$y_1 = re^{ax}.\sin(bx+c+\theta)$ —— (ii)

Again differentiating Eqn (ii)

$$y_2 = r\left[e^{ax} \cdot b\cos(bx+c+\theta) + e^{ax} \cdot a\sin(bx+c+\theta)\right]$$

$$= re^{ax}\left[a\sin(bx+\theta+c) + b\cos(bx+\theta+c)\right]$$

$$= re^{ax}\left[r\cos\theta \cdot \sin(bx+c+\theta) + r\sin\theta\cos(bx+c+\theta)\right]$$

$$= r^2e^{ax}\left[\cos\theta \cdot \sin(bx+c+\theta) + \sin\theta\cos(bx+c+\theta)\right]$$

$$= r^2 \cdot e^{ax} \cdot \sin(bx+c+\theta+\theta)$$

$$y_2 = r^2 \cdot e^{ax} \cdot \sin(bx+c+2\theta) \text{ — (iii)}$$

Again and Again diff. then

$$\boxed{y_n = r^n \cdot e^{ax} \cdot \sin(bx+c+n\theta)} \text{ — (iv)}$$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$

Differentiating Eqn (iv) w.r.to x we get

$$y_{n+1} = r^n\left[ae^{ax} \cdot \sin(bx+c+n\theta) + e^{ax} \cdot b\cos(bx+c+n\theta)\right]$$

$$= r^n\left[r\cos\theta\sin(bx+c+n\theta) + r\sin\theta\cos(bx+c+n\theta)\right]e^{ax}$$

$$= r^{n+1} \cdot e^{ax}\left[\cos\theta \cdot \sin(bx+c+n\theta) + \sin\theta\cos(bx+c+n\theta)\right]$$

$$= r^{n+1}e^{ax}\left[\sin(bx+c+n\theta+\theta)\right] = r^{n+1}e^{ax}\sin(bx+c+\theta(n+1))$$

$$y_{n+1} = r^{n+1}e^{ax}\left[\sin(bx+c+\theta(n+1))\right] \text{ — (5) (v)}$$

Which is the same form as Eqn (4) so Eqn (5) is true for $n+1$

Since Eqn (v) is true for $n = 1, 2, 3 \cdots$ so on.
therefore by mathematical induction method is true for all n.

Thus. $y = e^{ax}\sin(bx+c)$ then

$$y_n = r^n e^{ax}\sin(bx+c+n\theta)$$

where $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1} b/a$

$$y_n = (a^2+b^2)^{n/2}\cdot\left[\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)\right]$$

Similarly, $y = e^{ax}\cos(bx+c)$ then

$$y_n = r^n e^{ax}\cos(bx+c+n\theta)$$

where $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1} b/a$