Exp. To find y_n if $y = \tan^{-1}\frac{x}{a}$

Solution:- Given

$$y = \tan^{-1}\frac{x}{a} \quad \text{--- (1)}$$

Now, differentiate Eqn (1) w.r.to x

$$y_1 = \frac{d}{dx}\tan^{-1}\frac{x}{a}$$

$$y_1 = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1 \cdot a^2}{a(a^2+x^2)}$$

$$y_1 = \frac{a}{a^2+x^2}$$

Partial fraction of $\frac{a}{a^2+x^2}$

$$y_1 = \frac{a}{a^2+x^2} = \frac{a}{(x+ia)(x-ia)}$$

$$= a\cdot\frac{1}{2ia}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

Then, $\frac{dy}{dx} = y_1 = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$ --- (2)

Differentiate Eqn (2) $(n-1)$ number of times, we get

$$\therefore \quad y_n = \frac{1}{2i}\frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$y_n = \frac{1}{2i}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x-ia}\right) - \frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x+ia}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{(-1)^{n-1}\lfloor n-1}{(x-ia)^n} - \frac{(-1)^{n}\lfloor n-1}{(x+ia)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i}\left[\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right]$$

Put $x = r\cos\theta$ & $a = r\sin\theta$

$r^2 = a^2 + x^2$ $\quad \theta = \tan^{-1}\frac{a}{x}$

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}\lfloor n-1}{2i\, r^n}\left[\frac{1}{(e^{-i\theta})^n} - \frac{1}{(e^{+i\theta})^n}\right]$$

$$= \frac{(-1)^{n-1}\lfloor n-1}{2i\, r^n}\cdot\left[e^{+in\theta} - e^{-in\theta}\right]$$

$$y_n = \frac{(-1)^{n-1}}{\cancel{2i}}\cdot\frac{\lfloor n-1}{r^n}\left[\cancel{2i}\sin n\theta\right]$$

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{r^n}\sin n\theta \quad \text{put } r = \frac{a}{\sin\theta}$$

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{a^n}\cdot(\sin\theta)^n\cdot\underline{\sin n\theta}$$