Exp. To find y_n if $y = \tan^{-1}x$

Sol. We have

$$y = \tan^{-1}x$$

Diff. w.r.to x bothsides, we get

$$y_1 = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{(x+i)-(x-i)}{(x+i)(x-i)}\right]$$

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right] \quad \text{--- (1)}$$

Diff. y_1 w.r.to x in (n-1) times, we get

$$y_n = \frac{1}{2i}\left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x-i}\right) - \frac{d^{n-1}}{dx^{n-1}}\left(\frac{1}{x+i}\right)\right] \quad \text{--- (2)}$$

We know as $y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$ if $y = \frac{1}{ax+b}$

applying in Eq (2)

Then

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}\lfloor(n-1)}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right] \quad \text{--- (3)}$$

Now put $x = r\cos\theta$ and $1 = r\sin\theta$

so that $r = \sqrt{x^2+1}$ & $\theta = \tan^{-1}\frac{1}{x}$

Then we have from Eqn (3)

$$y_n = \frac{(-1)^{n-1}\lfloor(n-1)}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1} \lfloor n-1}{2i} \cdot \frac{1}{r^n}\left[\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1} \cdot \lfloor n-1}{2i\, r^n}\left[\frac{1}{e^{-in\theta}} - \frac{1}{e^{in\theta}}\right] \quad \because e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_n = \frac{(-1)^{n-1} \lfloor n-1}{2i\, r^n}\left[e^{in\theta} - e^{-in\theta}\right]$$

$$y_n = \frac{(-1)^{n-1} \cdot \lfloor n-1}{2i\, r^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right]$$

$$y_n = \frac{(-1)^{n-1} \lfloor n-1}{\cancel{2i}\, r^n} \cancel{2i} \sin n\theta$$

$$y_n = \frac{(-1)^{n-1} \lfloor n-1}{r^n} \cdot \sin n\theta$$

put $r = \frac{a}{\sin\theta}$

$$\Rightarrow y_n = \frac{(-1)^{n-1} \cdot \lfloor n-1}{a^n} \cdot \sin^n\theta \cdot \sin n\theta$$

$$y_n = (-1)^{n-1} (\lfloor n-1) \cdot \sin^n\theta \cdot \sin n\theta$$