Derivation of Magnon or ferromagnetic spin wave Dispersion Relation :To obtain a relation between frequency of spin waves and lattice constant , we consider the spin S_i at the site i and it is coupled with nearest neighbour spins S_{i-1} and s_{i+1} . H is an effective magnetic field on the spin , a is lattice constant and ω is frequency of spin waves .

Semi-classical derivation

Linear chain of aligned spins (S=Sz, Sx=Sy=0)

Small perturbation Sz~S, Sx,Sy<<S

Spins treated as classical vectors

Time dependence of expected value of <Sj>

$$\hat{\mathcal{H}} = -2\mathrm{J}\sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1}$$

$$
\begin{aligned}
\frac{\mathrm{d}S_j^x}{\mathrm{d}t} &\approx \quad \frac{2JS}{\hbar}(2S_j^y - S_{j-1}^y - S_{j+1}^y) \\
\frac{\mathrm{d}S_j^y}{\mathrm{d}t} &\approx -\frac{2JS}{\hbar}(2S_j^x - S_{j-1}^x - S_{j+1}^x) \\
\frac{\mathrm{d}S_j^z}{\mathrm{d}t} &\approx 0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\langle\hat{\mathbf{S}}_j\rangle}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}\langle[\hat{\mathbf{S}}_j, \hat{\mathcal{H}}]\rangle \\
&= -\frac{2\mathrm{J}}{\mathrm{i}\hbar}\langle[\hat{\mathbf{S}}_j, \cdots + \hat{\mathbf{S}}_{j-1}\cdot\hat{\mathbf{S}}_j + \hat{\mathbf{S}}_j\cdot\hat{\mathbf{S}}_{j+1} + \cdots]\rangle \\
&= -\frac{2\mathrm{J}}{\mathrm{i}\hbar}\langle[\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_{j-1}\cdot\hat{\mathbf{S}}_j] + [\hat{\mathbf{S}}_j, \hat{\mathbf{S}}_j\cdot\hat{\mathbf{S}}_{j+1}]\rangle \\
&= \frac{2\mathrm{J}}{\hbar}\langle\hat{\mathbf{S}}_j \times (\hat{\mathbf{S}}_{j-1} + \hat{\mathbf{S}}_{j+1})\rangle.
\end{aligned}
$$

$$S_j^x = A e^{i(qja - \omega t)}$$

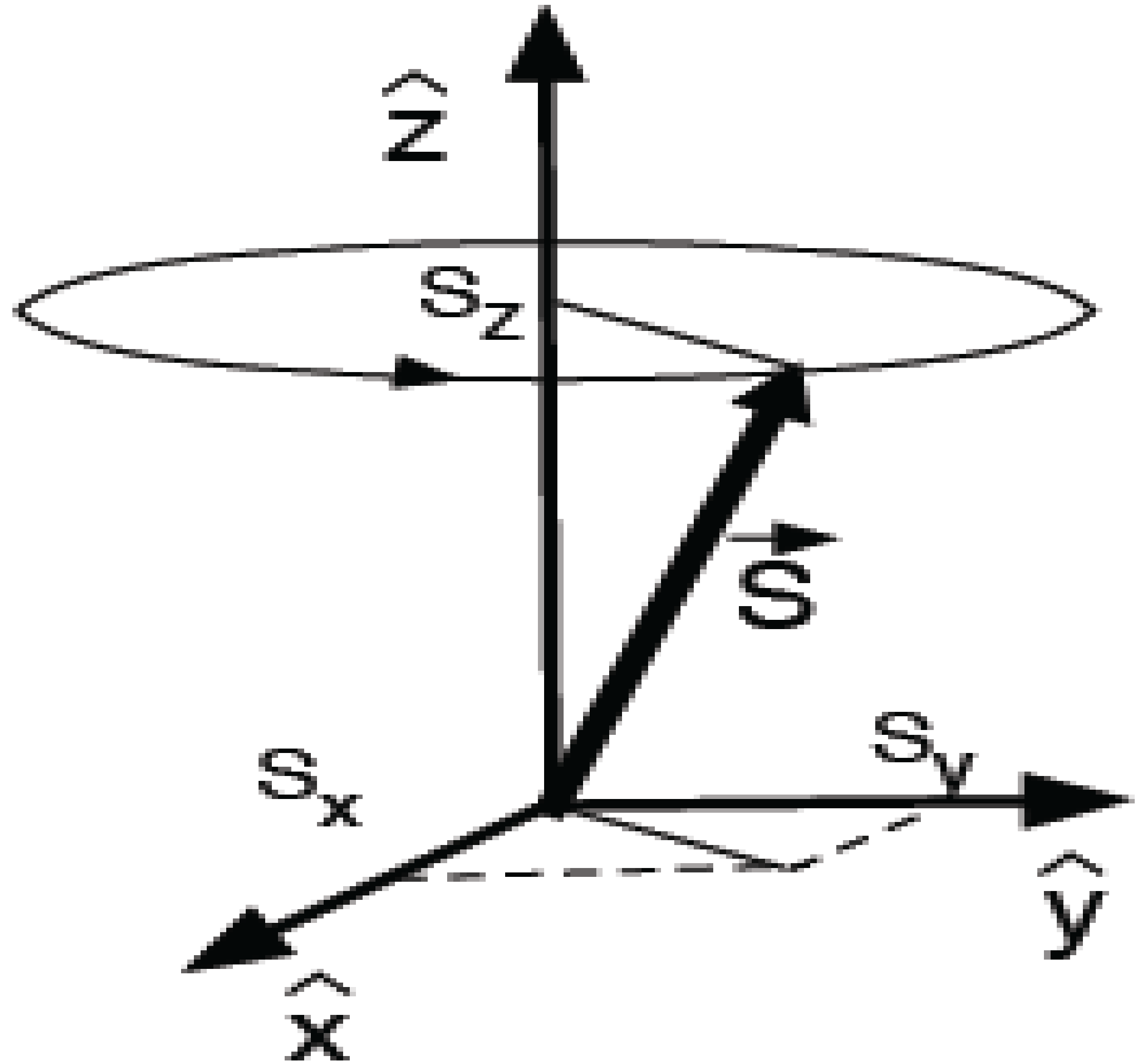
$$S_j^y = B e^{i(qja - \omega t)}$$

Raising and lowering operators are defined as:

$$\hat{S}^+ = \hat{S}^x + i\hat{S}^y$$

$$\hat{S}^- = \hat{S}^x - i\hat{S}^y$$

$$
\begin{aligned}
\hat{S}^{+} \mid \uparrow_z \rangle &= 0 \\
\hat{S}^{+} \mid \downarrow_z \rangle &= \mid \uparrow_z \rangle \\
\hat{S}^{-} \mid \uparrow_z \rangle &= \mid \downarrow_z \rangle \\
\hat{S}^{-} \mid \downarrow_z \rangle &= 0
\end{aligned}
$$



ẑ
Sz
S⃗
Sy
Sx
ŷ
x̂

Commutation relations

$$\begin{aligned}
\left[\hat{S}^{+}, \hat{S}^{-}\right] &= 2\hat{S}^{z} \\
\left[\hat{S}^{z}, \hat{S}^{\pm}\right] &= \pm\hat{S}^{\pm} \\
\left[\hat{\mathbf{S}}^{2}, \hat{S}^{\pm}\right] &= 0
\end{aligned}$$

$$\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \hat{S}_z^2 + \frac{1}{2}\left(\hat{S}^+\hat{S}^- + \hat{S}^-\hat{S}^+\right)$$

$$\hat{S}^+\hat{S}^- + \hat{S}^-\hat{S}^+ = 2\left[(\hat{S}^x)^2 + (\hat{S}^z)^2\right]$$