Linear Differential Equation

Exp. Solve $\frac{d^2y}{dx^2} - 4y = x\sinh x$

Solution Given equation in symbolic form

$$(D^2-4)y = x\sinh x$$

To find C.F from $(D^2-4)y=0$

The auxiliary Equation is $D^2-4=0$

$$\Rightarrow D = \pm 2$$

Thus $C.F = C_1 e^{2x} + C_2 e^{-2x}$

To find P.I for $x.\sinh x$

$$P.I = \frac{1}{D^2-4}.x.\sinh x$$

$$= \frac{1}{(D^2-4)} x.\left(\frac{e^x - e^{-x}}{2}\right) \quad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}x(e^x - e^{-x})\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}.e^x.x - \frac{1}{D^2-4}e^{-x}.x\right]$$

$$= \frac{1}{2}\left[e^x.\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}\right]$$

$$\left[\because \frac{1}{f(D)}x.e^x = e^x\frac{1}{f(D+1)}\right]$$

$$= \frac{1}{2}\left[e^x.\frac{1}{D^2+1+2D-4}x - e^{-x}\frac{1}{D^2+1-2D-4}x\right]$$

$$P.I = \frac{1}{2}\left[e^x.\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$P.I = \frac{1}{2}\left[e^{x} \cdot \frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)} x - e^{2x} \cdot \frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)} x \right]$$

$$= \frac{1}{2}\left[\frac{e^{x}}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} x \right]$$

$$= -\frac{1}{6}\left[e^{x}\left(1+\frac{2D}{3}+\cdots\right)x - e^{-x}\left(1-\frac{2D}{3}+\cdots\right)x \right]$$

$\because (1-x)^{-1} = 1+x+x^2+x^3+\cdots$

$\because (1+x)^{-1} = 1-x+x^2-x^3+\cdots$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\frac{d}{dx}(x)+0\cdots\right) - e^{-x}\left(x-\frac{2}{3}\frac{d}{dx}(x)+0-\right)\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\cdot 1\right) - e^{-x}\left(x-\frac{2}{3}\cdot 1\right)\right]$$

$$= -\frac{1}{6}\left[e^{x}.x + e^{x}.\frac{2}{3} - e^{-x}x + e^{-x}.\frac{2}{3}\right]$$

$$= -\frac{1}{3}\left[x\left(\frac{e^{x}-e^{-x}}{2}\right) + \frac{2}{3}\left(\frac{e^{x}+e^{-x}}{2}\right)\right]$$

$$= -\frac{1}{3}\left[x.\sinh x + \frac{2}{3}\cosh x\right]$$

$$P.I = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the G.S is

$$y = 9e^{2x} + C_1 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$