Exp. Which of the following functions f, define on vectors $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$ on R^2 are bilinear forms.

(I) $f(\alpha, \beta) = x_1 y_2 - x_2 y_1$

(II) $f(\alpha, \beta) = x_1 y_1 + x_2 y_2$

(III) $f(\alpha, \beta) = (x_1 - y_1)^2 + x_2 y_2$

Sol. Let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$ and $\gamma = (z_1, z_2)$ be three vectors in R^3.

Let $a, b, c \in F(R)$

Then $a\alpha + b\beta = a(x_1, x_2) + b(y_1, y_2)$

$$a\alpha + b\beta = (ax_1 + by_1, ax_2 + by_2)$$

(I) By definition of bilinear form (f)

$$f(\alpha, \gamma) = x_1 z_2 - x_2 z_1$$

$$f(\beta, \gamma) = y_1 z_2 - y_2 z_1$$

How $f(\gamma, \alpha) = z_1 x_2 - x_1 z_2$

$$f(\gamma, \beta) = z_1 y_2 - y_1 z_2$$

(i) $f(a\alpha + b\beta, \gamma) = f\{(ax_1 + by_1, ax_2 + by_2), (z_1, z_2)\}$

$$= \{(ax_1 + by_1)z_2 - (ax_2 + by_2)z_1\}$$

$$= \{ax_1 z_2 + by_1 z_2 - ax_2 z_1 - by_2 z_1\}$$

$$= \{a(x_1 z_2 - x_2 z_1) + b(y_1 z_2 - y_2 z_1)\}$$

$$= af(\alpha, \gamma) + bf(\beta, \gamma)$$

$$f(a\alpha + b\beta, \gamma) = af(\alpha, \gamma) + b(\beta, \gamma)$$

Also $f(\gamma, a\alpha + b\beta) = f\{(z_1, z_2), (ax_1 + by_1, ax_2 + by_2)\}$

$= z_1(ax_2 + by_2) - z_2(ax_1 + by_1)$

$= a(z_1x_2 - z_2x_1) + b(z_1y_2 - y_1z_2)$

$f(\gamma, a\alpha + b\beta) = af(\gamma, \alpha) + bf(\gamma, \beta)$

Hence f is linear form on R^2.

(ii) By definition of f. where

$f(\alpha, \gamma) = x_1z_1 + x_2z_2$

$f(\beta, \gamma) = y_1z_1 + y_2z_2$

$f(\gamma, \alpha) = z_1x_1 + z_2x_2$

$f(\gamma, \beta) = z_1y_1 + z_2y_2$

How $f(a\alpha + b\beta, \gamma) = f\{(a(x_1, x_2) + b(y_1, y_2)), (z_1, z_2)\}$

$= f\{(ax_1 + by_1, ax_2 + by_2), (z_1, z_2)\}$

$= (ax_1 + by_1)z_1 + (ax_2 + by_2)z_2$

$= a(x_1z_1 + x_2z_2) + b(y_1z_1 + y_2z_2)$

$= af(\alpha, \gamma) + bf(\beta, \gamma)$

Also $f(\gamma, a\alpha + b\beta) = f((z_1, z_2), (ax_1 + by_1, ax_2 + by_2)\}$

$= z_1(ax_1 + by_1) + z_2(ax_2 + by_2)$

$= a(z_1x_1 + z_2x_2) + b(z_1y_1 + z_2y_2)$

$= af(\gamma, \alpha) + bf(\gamma, \beta)$

Hence f is bilinear form on R^2

(iii) Here $f(a\alpha + b\beta, \gamma) = f((ax_1 + by_1, (ax_2 + by_2), (z_1, z_2))$

$= (ax_1 + by_1 - z_1)^2 + (ax_2 + by_2)z_2$

Also of $af(\alpha, \gamma) + bf(\beta, \gamma) = a\{(x_1 - z_1)^2 + x_2z_2\} + b\{(y_1 - z_1)^2 + y_2z_2\}$

is not $\neq f(a\alpha + b\beta, \gamma)$ Hence f is not bilinear form