Linear differential Equation

Working procedure to solve the differential Equation:-

Let a diff. Eqn

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \quad k_{n-1} \frac{dy}{dx} + k_n y = X$$

of which the symbolic form is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots k_{n-1} D + k_n) y = X$$

Step-1:- To find the complemetary function (C.F)

(i) Write the auxilary Equation (A.E)

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots k_{n-1} D + k_n = 0$$

and solve it for D.

(ii) Write the C.F as follows:-

Roots of A.E	C.F
(i) $m_1, m_2, m_3 \cdots$ (real & distinct roots)	$\Rightarrow c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \cdots$
(ii) $m_1, m_2 = m_1, m_3 \cdots$ (two real & equal)	$\Rightarrow (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots$
(iii) $\alpha + i\beta, \alpha - i\beta, m_3 \cdots$ (a pair of imaginary roots & real roots others)	$\Rightarrow e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots$
(iv) $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \cdots$ (2 pairs of equal imaginary roots)	$\Rightarrow$ C.F: $e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \cdots$

Step-2: To find the particular Integral (P.I)

$$P.I = \frac{1}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots k_{n-1} D - k_n} X$$

$$= \frac{1}{f(D)} \text{ or } \frac{1}{\phi(D^2)} X.$$

(i) When $X = e^{ax}$

$$P.I = \frac{1}{f(D)} \cdot e^{ax}, \quad \text{put } D = a \quad \text{if } [f(a) \neq 0$$

$$= x \cdot \frac{1}{f'(D)} e^{ax}, \quad \text{put } D = a \quad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \quad \text{put } D = a \quad [f'(a) = 0, f''(a) \neq 0$$

and so on. $f'(D)$ = differentiating w.r.t. D

$f''(D)$ = diff. $f'(D)$ w.r.t. D

(ii) When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{1}{\phi(D^2)} \sin(ax+b) \quad \text{put } D^2 = -a^2 \quad [\phi(-a^2) \neq 0]$$

$$= x \cdot \frac{1}{\phi'(D^2)} \sin(ax+b) \quad \text{put } D^2 = -a^2$$

$[\phi(-a^2) = 0, \phi'(-a^2) \neq 0$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax+b) \quad \text{put } D^2 = -a^2$$

$[\phi(-a^2) = 0, \phi'(-a^2) = 0, \phi''$ [illegible]

and so on. $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. [illegible]

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. [illegible]

(iii) When $X = x^m$, m is +ve integer

$$P.I = \frac{1}{f(D)} x^m$$

$$= [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending power of D by binomial theorem.

$\Rightarrow$ B.T. $(1+D)^m = 1 + \frac{m}{\underline{|1}} D + \frac{m(m-1)}{\underline{|2}} D^2 + \dots \frac{m(m-1)(m-2)\dots}{\underline{|m}}$

(iv) when X is any function of x.

$$P.I = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$$

Step-3:- To find the complete solution

Then the $y = C.F + P.I$

Here

$$\Rightarrow P.I = \frac{1}{f(D)} X$$

$$f(D) = (D-m_1)(D-m_2) \cdots (D-m)$$

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \frac{A_3}{D-m_3} + \cdots \frac{A_n}{D-m_n}$$

$$P.I = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots - \frac{A_n}{D-m_n}\right] X$$

$$= \frac{A_1}{D-m_1} X + \frac{A_2}{D-m_2} X + \cdots \frac{A_n}{D-m_n} X$$

$$P.I = A_1 . e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \cdots$$
$$A_n e^{m_n x} \int X e^{-m_n x} dx$$