Definition - Let W be a subspace of vector space $V(F)$. Then a subspace W' of V such that

$$V = W \oplus W'$$ is called a complement of W

Theorem - If W is a subspace of a finite dimensional vector space $V(F)$ then there exists a subspace W' of V such that $V = W \oplus W'$

Proof Since W is a subspace of a FDVS, W is also finite dimensional let $\dim W = m$

Let $S_1 = \{w_1, w_2, \dots w_n\}$ be a basis of W, so that

$$L(S_1) = W$$

Now $w_1, w_2, \dots w_m$ being L.I in W are also L.I in V and so they can be extended to form a basis of V.

Let $S_2 = \{w_1, w_3 \dots w_m, v_1, v_3 \dots v_n\}$ be basis of V

Let $W' = L(S_3)$ where $S_2 = \{v_1, v_3 \dots v_n\}$.

Then W' is a subspace of V.

Let $v \in V$ be arbitrary.

Since S_2 is a basis of V, we can write

$$v = \alpha_1 w_1 + \alpha_2 w_2 + \dots \alpha_m w_m + \beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n \quad \text{—(1)}$$

$\alpha_i, \beta_j \in F$

In the above expression, $i = 1$ to m

$\alpha_1 w_1 + \alpha_2 w_2 + \dots \alpha_m w_m \in L(S_1) = W$ $j = 1$ to n

and $\beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n \in L(S_3) = W'$

Consequently by (1) $v \in W + W'$ $\forall v \in V$

$$V = W + W' \quad \text{— (2)}$$

Now we shall prove that $W \cap W' = \{0\}$

Let $x \in W \cap W'$ be arbitrary. Then $x \in W$ and $x \in W'$

Since S_1 is a basis of W

$x = \gamma_1 w_1 + \gamma_2 w_2 + \cdots \gamma_m w_m, \quad \gamma_i \in F$ — (3)

$i = 1$ to m

Since $W' = L(S_3)$, $x = \delta_1 v_1 + \delta_2 v_2 + \cdots \delta_n v_n$,

$\therefore \gamma_1 w_1 + \gamma_2 w_2 + \cdots \gamma_m w_m = \delta_1 v_1 + \delta_2 v_2 + \cdots \delta_n v_n, \quad \delta_j \in F$

$j = 1$ to n

$\gamma_1 w_1 + \gamma_2 w_2 + \cdots \gamma_m w_m - \delta_1 v_1 - \delta_2 v_2 - \cdots \delta_n v_n = 0$

$\gamma_1 = \gamma_2 = \cdots \gamma_m = 0, \quad \delta_1 = \delta_2 = \cdots = \delta_n = 0$ — (4)

Since S_2 is a basis of V.

From (3) & (4) $\quad x = 0$ and so $W \cap W' = \{0\}$ — (5)

From (2) and (5) $\quad V = W \oplus W'$

Hence proved.

Theorem Every subspace of a finite dimensional vector space V has a complement. Further, if W' is a complement of W, then

$$\dim W' = \dim V - \dim W$$

Proof Since W' is a complement of W we know $V = W \oplus W'$

This implies $\dim V = \dim W + \dim W'$

$\dim W' = \dim V - \dim W$

Hence proved