Exp. Show that the set $Q(\sqrt{2}) = \{a + b\sqrt{2} ; a, b \in Q\}$ is a vector space over Q (rational number) w.r.to the compositions:

(i) $(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2}$

(ii) $\alpha(a+b\sqrt{2}) = \alpha a + \alpha b\sqrt{2}$, $\alpha \in Q$.

Solution - The set $Q\sqrt{2} = \{a+b\sqrt{2} : a, b \in Q\}$ is an abelian group w.r.to the compositions:-

(i) $(a+b\sqrt{2}) + (c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2}$

The additive identity of $(Q\sqrt{2})$ is $0 = 0+0\sqrt{2}$

$(a+b\sqrt{2}) + 0+0\sqrt{2} = 0+0\sqrt{2} + a+b\sqrt{2} = a+b\sqrt{2}$ $\forall a, b \in Q$

and the additive inverse of $a+b\sqrt{2}$ is $-a-b\sqrt{2}$ then

$$a+b\sqrt{2} + (-a-b\sqrt{2}) = 0$$

Hence, $(Q\sqrt{2}, +)$ is abelian group.

Now, properties of scalar multiplication.

Let $\alpha, \beta \in Q$ and $u = a+b\sqrt{2}$, $v = c+d\sqrt{2}$ $\forall u, v \in Q\sqrt{2}$

From (ii) $\alpha(a+b\sqrt{2}) = \alpha a + \alpha b\sqrt{2} \in Q\sqrt{2}$

1. $\alpha(u+v) = \alpha((a+b\sqrt{2}) + (c+d\sqrt{2}))$
$= \alpha((a+c) + (b+d)\sqrt{2})$
$= \alpha(a+c) + \alpha(b+d)\sqrt{2}$
$= \alpha a + \alpha c + (\alpha b + \alpha d)\sqrt{2}$
$= \alpha(a+b\sqrt{2}) + \alpha(c+d\sqrt{2})$
$= \alpha u + \alpha v$

$\therefore \alpha(u+v) = \alpha u + \alpha v$

2. $(\alpha+\beta)u = (\alpha+\beta)(a+b\sqrt{2})$
$= (\alpha+\beta)a + (\alpha+\beta)b\sqrt{2}$
$= \alpha a + \beta a + (\alpha b + \beta b)\sqrt{2}$
$= \alpha(a+b\sqrt{2}) + \beta(a+b\sqrt{2})$
$= \alpha u + \beta u$

$\therefore (\alpha+\beta)u = \alpha u + \beta u$

3. $\alpha(\beta u) = \alpha(\beta a + \beta b\sqrt{2})$
$= (\alpha\beta)a + (\alpha\beta)b\sqrt{2}$
$= (\alpha\beta)(a + b\sqrt{2})$
$= (\alpha\beta)u$
$\therefore \alpha(\beta u) = (\alpha\beta)u$

4. $1 \cdot u = 1(a+b\sqrt{2}) = a+b\sqrt{2} = u \quad \forall u \in Q\sqrt{2}$

Hence, $Q(\sqrt{2})$ is a vector space over Q.

Exp. The set $M_2 = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in R\right\}$ is a vector space over R (real number) w.r.to matrix addition and multiplication of a matrix by a scalar (α) i$\alpha \in R$

Solution We know M_2 is a abelian group w.r.t. matrix addition.

Further $\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} \in M_2, \alpha \in R$

Let $\alpha, \beta \in R$ and $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$. Then $\alpha A \in M_2$ and

$A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

$\alpha(A+B) = \begin{bmatrix} \alpha(a_1+a_2) & \alpha(b_1+b_2) \\ \alpha(c_1+c_2) & \alpha(d_1+d_2) \end{bmatrix}$

(i) $\alpha(A+B) = \alpha A + \alpha B$

(ii) $(\alpha+\beta)A = \alpha A + \beta A$

(iii) $\alpha(\beta A) = (\alpha\beta)A$

(iv) $1A = A$

Hence, M_2 is a vector space over R.