Q.) Define vector triple product of vectors. prove that

(i) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a}\cdot\vec{c}\,\vec{b} - \vec{a}\cdot\vec{b}\,\vec{c}$

(ii) $\vec{a} \times \vec{b} \times \vec{c} = \vec{a}\cdot\vec{c}\,\vec{b} - \vec{b}\cdot\vec{c}\,\vec{a}$

(iii) cross product of vector is not associative.

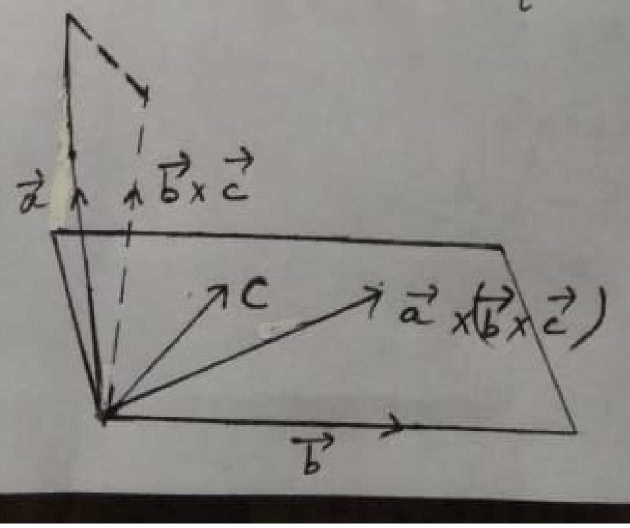
Definition :-

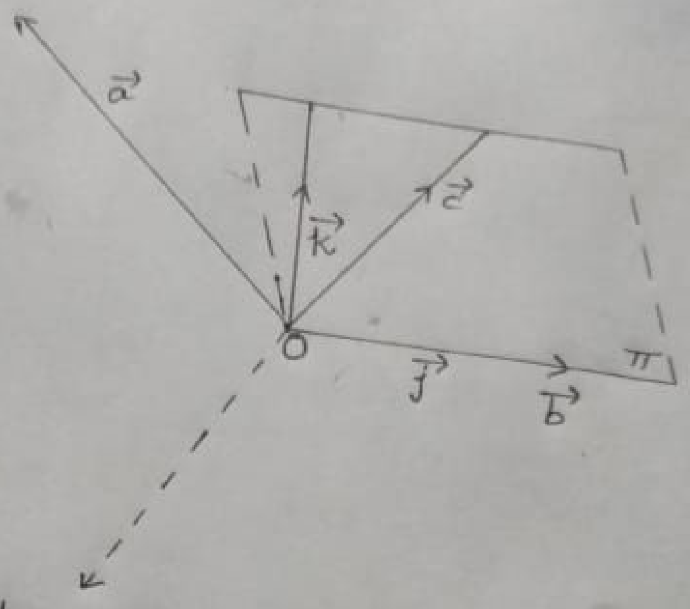
Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. Then the vector product of any one vector with the vector product of the other two vectors is called the vector triple product of $\vec{a}, \vec{b}, \vec{c}$.

Thus, $\vec{a} \times (\vec{b} \times \vec{c}), (\vec{a} \times \vec{b}) \times \vec{c}$ etc. are vector triple products of $\vec{a}, \vec{b}, \vec{c}$.

(i) we note that if the plane containing $\vec{b}$ and $\vec{c}$ the π then $\vec{b} \times \vec{c}$ is a vector $\vec{l}$ perpendicular to π. Also the product $\vec{a} \times (\vec{b} \times \vec{c})$ i.e. $\vec{a} \times \vec{l}$ is perpendicular to the plane $\vec{a}$ and l.

Hence $\vec{a} \times \vec{l}$ lies in the plane π. i.e. $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane of b and $\vec{c}$.





Let $\vec{j}$ be the unit vector along $\vec{b}$ and $\vec{k}$ to be the unit vector perpendicular to $\vec{b}$ and lying in the plane, of $\vec{b}$ and $\vec{c}$. Let $\vec{i}$ be the unit vector so that $\vec{i}, \vec{j}, \vec{k}$ form a right handed system of orthogonal triads.

Then we can put

$$\vec{b} = b_2 \vec{j}$$

$$\vec{c} = c_2 \vec{j} + c_3 \vec{k}$$

and

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\therefore \vec{b} \times \vec{c} = b_2 c_3 \vec{i}$$

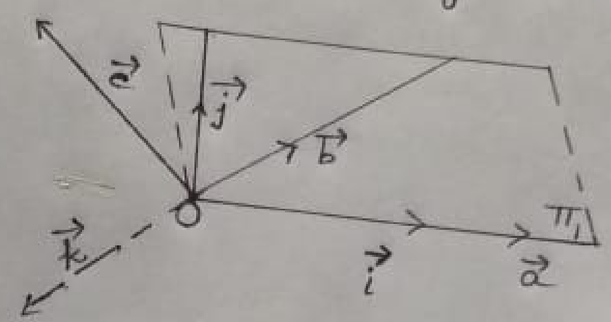
$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = -a_2 b_2 c_3 \vec{k} + a_3 b_2 c_3 \vec{j}$$

$$= -a_2 b_2 c_3 \vec{k} + a_3 b_2 c_3 \vec{j} + a_2 c_2 b_2 \vec{j} - a_2 c_2 b_2 \vec{j}$$

$$= (a_2 c_2 + a_3 c_3) b_2 \vec{j} - a_2 b_2 (c_2 \vec{j} + c_3 \vec{k})$$

$$= \vec{a} \cdot \vec{c}\, \vec{b} - \vec{a} \cdot \vec{b}\, \vec{c}$$

(ii) we note that if the plane containing
$\vec{a}$ and $\vec{b}$ be π_1, then $\vec{a} \times \vec{b}$ is a
vector $\vec{m}$ perpendicular to π_1. Also
the product $(\vec{a} \times \vec{b}) \times \vec{c}$.
i.e. $\vec{m} \times \vec{c}$ is perpendicular to
the plane of $\vec{m}$ and $\vec{c}$. Hence $\vec{m} \times \vec{c}$
lies in the plane π_1,
i.e. $(\vec{a} \times \vec{b}) \vec{c}$ lies
in the plane of $\vec{a}$ and $\vec{b}$



Let $\vec{i}$ be the unit vector along $\vec{a}$
and $\vec{j}$ the unit vector perpendicular
to $\vec{a}$ and lying in the plane of
$\vec{a}$ and $\vec{b}$. Let $\vec{k}$ be the unit vector
so that $\vec{i}, \vec{j}, \vec{k}$ form a right
handed system of orthogonal traids,
then we can write

$$\vec{a} = a_1 \vec{i}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j}$$

$$\vec{c} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k}$$

$$\therefore \vec{a} \times \vec{b} = a_1 b_2 \vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = a_1 b_2 c_1 \vec{j} - a_1 b_2 c_2 \vec{i}$$

$$= a_1 b_2 c_1 \vec{j} - a_1 b_2 c_2 \vec{i} + a_1 c_1 b_1 \vec{i} - a_1 c_1 b_1 \vec{i}$$

$$= a_1 c_1 (b_1 \vec{i} + b_2 \vec{j}) - (b_1 c_1 + b_2 c_2) a_1 \vec{i}$$

$$= \vec{a} \cdot \vec{c}\, \vec{b} - \vec{b} \cdot \vec{c}\, \vec{a}$$

(iii) From (i) and (ii) we find that

$$\vec{a} \times \vec{b} \times \vec{c} \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

This shows that vector product of vectors is not associative.

Remark :—

If $\vec{a}, \vec{b}, \vec{c}$ are non-zero vectors then $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

only when $\vec{a} \cdot \vec{c}\, \vec{b} - \vec{a} \cdot \vec{b}\, \vec{c} = \vec{a} \cdot \vec{c}\, \vec{b} - \vec{b} \cdot \vec{c}\, \vec{a}$

i.e. only when $\vec{a} \cdot \vec{b}\, \vec{c} = \vec{b} \cdot \vec{c}\, \vec{a}$

i.e. only when $\vec{c}$ is parallel to (Collinear) with $\vec{a}$.