Vector Spaces over a Field, V(F) : Let V be a non-empty set and F is a field, such that V is a vector space over F, If the following conditions/properties/axioms are satisfied.

1- $u+v \in V$, $\forall$ $u \in V$ and $v \in V$

2. $(u+v)+w = u+(v+w)$ $\forall$ $u, v, w \in V$

3. $u+v = v+u$, for all ($\forall$) $u\, v \in V$

4. There exists an element $e = 0$ in V, such that $v+0 = 0+v = v$, $\forall$ $v \in V$, $0 \in V$ is called zero vector in V.

5. For each $u \in V$, there exists some $v \in V$ such that $u+v = v+u = 0$. v as $-u$. Thus

$$u+(-u) = (-u)+u = 0$$

$-u$ is called the negative or additive inverse of u

6. $\alpha(u+v) = \alpha u + \alpha v$, for all $\alpha \in F$ & $u, v \in V$

7. $(\alpha+\beta)v = \alpha v + \beta v$, $\forall$ $\alpha, \beta \in F$ & $v \in V$

8. $\alpha(\beta v) = (\alpha\beta)v$, $\forall$ $\alpha, \beta \in F$ & $v \in V$

9. If 1 is unity of F, then $1v = v$ for all $v \in V$

(*) A vector space V over a field F is denoted by V(F). The elements of V are called vectors and the elements of F are called scalars. Vectors will be written as u, v, w, x, y, z and scalars as α, β, γ, a, b, c etc.

I → The first 5 properties of a vector space imply that (V, +) is an abelian group.

II → The zero elements of V and F are both denoted by 0. $V = \{0, 0, 0, 0, 0\}$
$F = (0, 0, 0, 0, 0)$

A vector space (V) is a collection of objects called vectors, which may be added together and multiplied by real numbers (scalars).

⇒ $(V, +)$ is an abelian group, if following 1-5 properties are satisfied.

⇒ Properties of Scalar Multiplication

Let $\alpha, \beta \in R$ and $u = (a_1, a_2, a_3) \in V$,
$v = (b_1, b_2, b_3) \in V$

(i) $\alpha(u+v) = \alpha u + \alpha v$, $\forall\ u, v \in V\ \forall\ \alpha \in R$

(ii) $(\alpha+\beta)u = \alpha u + \beta u$ $\forall\ u \in V, \alpha, \beta \in R$

(iii) $\alpha(\beta u) = (\alpha\beta)u$ $\forall\ u \in V\ \&\ \alpha, \beta \in R$

(iv) $1 \cdot u = u$ $\forall\ u \in V\ \forall\ 1 \in R$

<u>Exp.</u> Let $V = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in R\} = R^3$ be the set of all 3-tuples (dimentions) of real numbers. We define the addition and scalar multiplication in V as follows:-

$A \to (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3)$

$B \to \alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$

Then V is a vector space over the field R of real numbers.

<u>Solution</u> $(V, +)$ is an abelian group.

Let $u = (a_1, a_2, a_3) \in V$
$v = (b_1, b_2, b_3) \in V$
$w = (c_1, c_2, c_3) \in V$

(i) $u + v \in V \Rightarrow (a_1, a_2, a_3) + (b_1, b_2, b_3)$

$u+v = (a_1+b_1, a_2+b_2, a_3+b_3)$

$u+v \in V$

(ii) $(u+v)+w = \{(a_1, a_2, a_3) + (b_1, b_2, b_3)\} + (c_1, c_2, c_3)$

$= (a_1+b_1, a_2+b_2, a_3+b_3) + (c_1, c_2, c_3)$

$= (a_1+b_1+c_1, a_2+b_2+c_2, a_3+b_3+c_3)$

$(u+v)+w = (a_1, a_2, a_3) + (b_1+c_1, b_2+c_2, b_3+c_3)$

$(u+v)+w = u+(v+w)$

(3) $u+v = (a_1, a_2, a_3) + (b_1, b_2, b_3)$

$= (a_1+b_1, a_2+b_2, a_3+b_3)$

$= (b_1+a_1, b_2+a_2, b_3+a_3)$

$= (b_1, b_2, b_3) + (a_1, a_2, a_3)$

$u+v = v+u$

(4) $u+0 = (a_1, a_2, a_3) + (0, 0, 0)$

$= (a_1+0, a_2+0, a_3+0)$

$= (0+a_1, 0+a_2, 0+a_3)$

$= (0, 0, 0) + (a_1, a_2, a_3)$

$u+0 = 0+u \Rightarrow (a_1, a_2, a_3) = u$

(5) $u+(-u) = (a_1, a_2, a_3) + (-a_1, -a_2, -a_3)$

$= (a_1-a_1, a_2-a_2, a_3-a_3)$

$= (0, 0, 0)$

$u+(-u) = 0$

Hence, $(V, +)$ is an abelian group.

Properties of scalar Multiplication

Let $\alpha, \beta \in R$ and

$u = (a_1, a_2, a_3) \in V$

$v = (b_1, b_2, b_3) \in V$

(1) $\alpha(u+v) = \alpha(a_1+b_1, a_2+b_2, a_3+b_3)$ by A

$= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3)$

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\alpha b_1, \alpha b_2, \alpha b_3)$

$= \alpha(a_1, a_2, a_3) + \alpha(b_1, b_2, b_3)$ by B

$= \alpha u + \alpha v$

$\therefore \alpha(u+v) = \alpha u + \alpha v$

(2) $(\alpha+\beta)u = (\alpha+\beta)(a_1, a_2, a_3)$

$= ((\alpha+\beta)a_1, (\alpha+\beta)a_2, (\alpha+\beta)a_3)$ by (B)

$= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2, \alpha a_3 + b_3 a_3)$
by distributionlaw

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\beta a_1, \beta a_2, \beta a_3)$

$= \alpha(a_1, a_2, a_3) + \beta(a_1, a_2, a_3)$

$= \alpha u + \beta u$

$\therefore (\alpha+\beta)u = \alpha u + \beta u$

(3) $\alpha(\beta u) = \alpha(\beta a_1, \beta a_2, \beta a_3)$

$= (\alpha\beta a_1, \alpha\beta a_2, \alpha\beta a_3)$

$= \alpha\beta(a_1, a_2, a_3)$ by (B)

$\alpha(\beta u) = (\alpha\beta)(u)$

(4) $1 \cdot u = (1a_1, 1a_2, 1a_3) = (a_1, a_2, a_3) = u \quad \forall u \in V$

Hence V is vectorspace over R