

(4)
(viii) Expansion formula

$$\text{Curl Curl } \vec{F} = \text{grad div } \vec{F} - \nabla^2 \vec{F}$$

$$= \text{grad div } \vec{F} - \sum \frac{\partial^2 \vec{F}}{\partial x^2}$$

Let $\vec{F} = \vec{i} F_1 + \vec{j} F_2 + \vec{k} F_3$

$$\therefore \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \vec{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$\therefore \text{Curl Curl } \vec{F}$

$$= \sum \vec{i} \times \frac{\partial}{\partial x} \left\{ \vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \vec{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right\}$$

$$= \sum \left\{ \vec{i} \times \vec{j} \frac{\partial}{\partial x} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \vec{i} \times \vec{k} \frac{\partial}{\partial x} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \right.$$

$$= \vec{k} \frac{\partial}{\partial x} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) - \vec{j} \frac{\partial}{\partial x} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= + \vec{i} \frac{\partial}{\partial y} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \vec{k} \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right)$$

$$+ \vec{j} \frac{\partial}{\partial z} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \vec{i} \frac{\partial}{\partial z} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right)$$

$$= \vec{i} \frac{\partial}{\partial x} \left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) + \vec{j} \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x} \right) + \vec{k} \frac{\partial}{\partial z} \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right)$$

$$- \left\{ \frac{\partial^2}{\partial x^2} (\vec{k} F_3 + \vec{j} F_2) + \frac{\partial^2}{\partial y^2} (\vec{i} F_1 + \vec{k} F_3) + \frac{\partial^2}{\partial z^2} (\vec{j} F_2 + \vec{i} F_1) \right\}$$

$$= \vec{i} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \vec{j} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \vec{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$\therefore \text{div curl } \vec{F}$

$$= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} \right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \right)$$

$= 0$