VECTOR CALCULUS

(4) (viii) Expansion formula

Curl Curl $\vec{F}$ = grad div $\vec{F} - \nabla^2 \vec{F}$

$$\left(= \text{grad div } \vec{F} - \sum \frac{\partial^2 \vec{F}}{\partial x^2}\right)$$

Let $\vec{F} = \vec{i} F_1 + \vec{j} F_2 + \vec{k} F_3$

$$\therefore \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \vec{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$\therefore$ curl Curl $\vec{F}$

$$= \sum \vec{i} \times \frac{\partial}{\partial x}\left\{\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \vec{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right\}$$

$$= \sum \left\{\vec{i} \times \vec{j} \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \vec{i} \times \vec{k} \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right.$$

$$= \vec{k} \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \vec{j} \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= + \vec{i} \frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \vec{k} \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)$$

$$+ \vec{j} \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{i} \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$

$$= \vec{i} \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \vec{j} \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x}\right) + \vec{k} \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)$$

$$- \left\{\frac{\partial^2}{\partial x^2}\left(\vec{k} F_3 + \vec{j} F_2\right) + \frac{\partial^2}{\partial y^2}\left(\vec{i} F_1 + \vec{k} F_3\right) + \frac{\partial^2}{\partial z^2}\left(\vec{j} F_2 + \vec{i} F_1\right)\right.$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$+ \vec{k}\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\Big] - \left\{\vec{i}\frac{\partial^2 F_1}{\partial x^2} + \vec{j}\frac{\partial^2 F_2}{\partial y^2}\right\}$$

$$- \left\{\frac{\partial^2}{\partial x^2}\left(\vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3\right) + \frac{\partial^2}{\partial y^2}\left(\vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3\right)\right.$$

$$\left. + \frac{\partial^2}{\partial z^2}\left(\vec{i}F_3 + \vec{j}F_2 + \vec{k}F_3\right)\right\}$$

$$+ \left\{\vec{i}\frac{\partial^2 F_1}{\partial x^2} + \vec{j}\frac{\partial^2 F_2}{\partial y^2} + \vec{k}\frac{\partial^2 F_3}{\partial z^2}\right\}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3\right)$$

$$= \text{grad div } \vec{F} - \nabla^2 \vec{F}$$

$$\left(= \text{grad div } \vec{F} - \sum \frac{\partial^2 \vec{F}}{\partial x^2}\right)$$

(ix) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then

(a) div $\vec{r} = 3$ (b) Curl $\vec{r} = 0$

(a) L.H.S. $= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}x + \vec{j}y + \vec{k}z\right)$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 3 = \text{R.H.S.}$$

(b) L.H.S. $= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \vec{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \vec{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

$$= 0 + 0 + 0$$

$$= \text{R.H.S.}$$