④ Expansion formula :-

(i) $\operatorname{div} \phi \vec{A} = \phi \operatorname{div} \vec{A} + \vec{A} \cdot \operatorname{grad} \phi$

Proof :-

L.H.S. $= \sum \vec{i} \cdot \frac{\partial}{\partial x} (\phi \vec{A})$

$= \sum \vec{i} \cdot \left\{ \frac{\partial \phi}{\partial x} \vec{A} + \phi \frac{\partial \vec{A}}{\partial x} \right\}$

$= \sum \vec{i} \frac{\partial \phi}{\partial x} \cdot \vec{A} + \phi \sum \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}$

$= (\operatorname{grad} \phi) \cdot \vec{A} + \phi \operatorname{div} \vec{A}$

$= \phi \operatorname{div} \vec{A} + \vec{A} \cdot \operatorname{grad} \phi$

= R.H.S.

(ii) $\operatorname{Curl} (\phi \vec{A}) = \phi \operatorname{Curl} \vec{A} - \vec{A} \times \operatorname{grad} \phi$

Proof :-

L.H.S. $= \sum \vec{i} \times \frac{\partial}{\partial x} (\phi \vec{A})$

$= \sum \vec{i} \times \left\{ \phi \frac{\partial \vec{A}}{\partial x} + \frac{\partial \phi}{\partial x} \vec{A} \right\}$

$= \phi \sum \vec{i} \times \frac{\partial \vec{A}}{\partial x} + \left(\sum i \frac{\partial \phi}{\partial x} \right) \times \vec{A}$

$= \phi \operatorname{Curl} \vec{A} + (\operatorname{grad} \phi) \times \vec{A}$

$= \phi \operatorname{Curl} \vec{A} - \vec{A} \times \operatorname{grad} \phi$

= R.H.S.

(iii) $\operatorname{div}(\vec{a}\times\vec{b}) = \vec{b}\cdot \text{Curl}\,\vec{a} - \vec{a}\cdot \text{curl}\,\vec{b}$

Proof :-

$$\text{L.H.S.} = \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$$

$$= \sum \vec{i}\cdot\left\{\frac{\partial\vec{a}}{\partial x}\times\vec{b} + \vec{a}\times\frac{\partial\vec{b}}{\partial x}\right\}$$

$$= \sum \vec{i}\times\frac{\partial\vec{a}}{\partial x}\cdot\vec{b} + \sum \vec{i}\cdot\vec{a}\times\frac{\partial\vec{b}}{\partial x}$$

[$\because$ in a scalar triple product dot and cross can be interchanged]

$$= \left(\sum \vec{i}\times\frac{\partial\vec{a}}{\partial x}\right)\cdot\vec{b} - \sum \vec{a}\cdot\vec{i}\times\frac{\partial\vec{b}}{\partial x}$$

$$= (\text{Curl}\,\vec{a})\cdot\vec{b} - \vec{a}\cdot\sum\vec{i}\times\frac{\partial\vec{b}}{\partial x}$$

$$= \vec{b}\cdot\text{curl}\,\vec{a} - \vec{a}\cdot\text{curl}\,\vec{b}$$

$= \text{R.H.S.}$

(iv) $\text{Curl}(\vec{a}\times\vec{b}) = (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b} + \vec{a}\operatorname{div}\vec{b} - \vec{b}\operatorname{div}\vec{a}$

Proof :-

We have $\text{curl}(\vec{a}\times\vec{b}) = \sum\vec{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$

$$= \sum\vec{i}\times\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right) + \sum\vec{i}\times\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right)$$

$$= \sum\left\{(\vec{i}\cdot\vec{b})\frac{\partial\vec{a}}{\partial x} - \left(\vec{i}\cdot\frac{\partial\vec{a}}{\partial x}\right)\vec{b}\right\}$$

$$+ \sum\left\{\left(\vec{i}\cdot\frac{\partial\vec{b}}{\partial x}\right)\vec{a} - (i\cdot\vec{a})\frac{\partial\vec{b}}{\partial x}\right\}$$

$$= \sum (\vec{b} \cdot \vec{i}) \frac{\partial \vec{a}}{\partial x} - (\operatorname{div} \vec{a}) \vec{b} + (\operatorname{div} \vec{b}) \vec{a} - \sum (\vec{a} \cdot \vec{i}) \frac{\partial \vec{b}}{\partial x}$$

$$= \sum \left(\vec{b} \cdot i \frac{\partial}{\partial x}\right) \vec{a} - \vec{b} \operatorname{div} \vec{a} + \vec{a} \operatorname{div} \vec{b} - \sum \left(\vec{a} \cdot i \frac{\partial}{\partial x}\right) \vec{b}$$

$$= (\vec{b} \cdot \nabla) \vec{a} - (\vec{a} \cdot \nabla) \vec{b} + \vec{a} \operatorname{div} \vec{b} - \vec{b} \operatorname{div} \vec{a}$$