Q(3) Define grad, div and curl of functions

Gradient of a scalar function

Let $\phi(x,y,z)$ be any scalar point function and $P(x,y,z)$ any point. Let Q be a point with $\vec{PQ} = s\vec{a}$ where $\vec{a}$ is the unit vector in the direction PQ and s = distance of Q from P.

Then $\lim_{s \to 0} \frac{\phi(P) - P(Q)}{s}$ is called the directional derivative of $\phi(x,y,z)$ in the direction $\vec{a}$ and is denoted by $\frac{d\phi}{ds}\vec{a}$. For different directions from P the values of the directional derivative is different. The directional derivative which has the maximum magnitude is called the gradient of ϕ at P and is denoted by grad ϕ.

It can be shown that

$$\text{grad } \phi = \nabla \phi$$

where $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Thus $$\text{grad } \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \sum i \frac{\partial \phi}{\partial x}, \text{ in short.}$$

Note :- It may be noted that grad ϕ is a vector and this vector is normal to the surface $\phi(x, y, z) = 0$ at $P(x, y, z)$.

Thus the unit normal to the surface $\phi(x, y, z) = 0$ at the point (x, y, z) is $\dfrac{\text{grad}\ \phi}{|\text{grad}\ \phi|}$.

Divergence of a vector function

Let $\vec{F}(x, y, z)$ be a vector point function. Then $\nabla \cdot \vec{F}$ i.e. $\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$ is called the divergence of $\vec{F}$ and is also denoted by div $\vec{F}$. Thus

$$\text{div}\ \vec{F} = \vec{i}\,\frac{\partial \vec{F}}{\partial x} + \vec{j}\,\frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

$$= \sum \vec{i}\,\frac{\partial \vec{F}}{\partial x}, \text{ in short.}$$

Moreover, if $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$, then

$$\text{div}\ \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note :- If div $\vec{F} = 0$, the vector function $\vec{F}$ is called a solenoidal vector function.

Curl (or rotation) of a vector function

Let $\vec{F}(x,y,z)$ be a vector point function.

Then $\nabla \times \vec{F}$ i.e. $\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{F}$ is called the curl (or rotation) of $\vec{F}$ and is also denoted by curl $\vec{F}$ (or rot. $\vec{F}$)

Thus

$$\text{Curl } \vec{F} = \vec{i}\times\frac{\partial \vec{F}}{\partial x} + \vec{j}\times\frac{\partial \vec{F}}{\partial y} + \vec{k}\times\frac{\partial \vec{F}}{\partial z}$$

$$= \sum \vec{i}\times\frac{\partial F}{\partial x} \text{, in brief}$$

Moreover, if $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$, then

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note :- If curl $\vec{F} = 0$, the vector function $\vec{F}$ is called an irrotational vector function.