(2) The necessary and sufficient condition that the vector function $\vec{u}(t)$ has a constant direction is that $\vec{u} \times \frac{d\vec{u}}{dt} = 0$

Proof :-

we have $\vec{u}(t) = u(t)\vec{g}(t)$ ——— (1)

where $|\vec{u}(t)| = u(t)$ and $g(t)$ is the unit vector in the direction of $\vec{u}(t)$.

Differentiating (1) $\frac{d\vec{u}}{dt} = u\frac{d\vec{g}}{dt} + \frac{du}{dt}\vec{g}$

$\therefore \vec{u} \times \frac{d\vec{u}}{dt} = ug \times \{u\frac{d\vec{g}}{dt} + \frac{d\vec{u}}{dt}\vec{g}\}$

$= u^2\vec{g} \times \frac{d\vec{g}}{dt} + u\frac{du}{dt}\vec{g} \times \vec{g}$

$\therefore \vec{u} \times \frac{d\vec{u}}{dt} = u^2\vec{g} \times \frac{d\vec{g}}{dt}$ ——— (2)

Necessity

Let $\vec{u}$ have a constant direction.

Then $g(\vec{t})$ = Constant = independent of t.

$\therefore \frac{d\vec{g}}{dt} = 0$

$\therefore$ (2) gives $\vec{u} \times \frac{d\vec{u}}{dt} = 0$

This proves the necessity.

Sufficiency:-

Let $\vec{u} \times \frac{d\vec{u}}{dt} = 0$

Then ② gives $u^2 \vec{g} \times \frac{d\vec{g}}{dt} = 0$

or, $\vec{g} \times \frac{d\vec{g}}{dt} = 0$ ——— ③

Now $\vec{g}(t)$ is of constant magnitude (it is a unit vector)

$\therefore \vec{g} \cdot \frac{d\vec{g}}{dt} = 0$ ——— ④

③ shows that $\frac{d\vec{g}}{dt}$ is collinear with $\vec{g}$ unless $\frac{d\vec{g}}{dt} = 0$ and ④ shows that $\frac{d\vec{g}}{dt}$ is perpendicular to $\vec{g}$ unless $\frac{d\vec{g}}{dt} = 0$ since a vector cannot be both collinear and perpendicular to another vector,

$\therefore$ ③ and ④ together show that $\frac{d\vec{g}}{dt} = 0$

or, $g(t) = \text{constant}$

i.e. $\vec{u}(t)$ has a constant direction.