

7.) Reciprocal system of vectors.

- (a) Define reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$.
- (b) If $\vec{a}', \vec{b}', \vec{c}'$ be reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$, prove that
- (i) $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}' = 0$
 - (ii) $\vec{a} \cdot \vec{a}' + \vec{b} \cdot \vec{b}' + \vec{c} \cdot \vec{c}' = 3$
 - (iii) $[\vec{a} \ \vec{b} \ \vec{c}] [\vec{a}' \ \vec{b}' \ \vec{c}'] = 1$
 - (iv) The set of vectors $\vec{a}, \vec{b}, \vec{c}$ is the reciprocal system of vectors $\vec{a}', \vec{b}', \vec{c}'$.
 - (v) Any vector \vec{n} can be expressed as
 - (I) $\vec{n} = \vec{n} \cdot \vec{a} \vec{a}' + \vec{n} \cdot \vec{b} \vec{b}' + \vec{n} \cdot \vec{c} \vec{c}'$
 - (II) $\vec{n} = \vec{n} \cdot \vec{a}' \vec{a} + \vec{n} \cdot \vec{b}' \vec{b} + \vec{n} \cdot \vec{c}' \vec{c}$
 - (III) $\vec{n} = \vec{n} \cdot \vec{i} \vec{i} + \vec{n} \cdot \vec{j} \vec{j} + \vec{n} \cdot \vec{k} \vec{k}$

sol (a) Definition :-

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar the vectors $\vec{a}', \vec{b}', \vec{c}'$ defined by

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]},$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

are called reciprocal to the vectors $\vec{a}, \vec{b}, \vec{c}$.

(2)

(F) (i) we have,

$$\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$$

$$= \frac{\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$= \frac{\{(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\} + \{(\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}\} + \{(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}\}}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$= \frac{\{(\vec{c} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\} + \{(\vec{a} \cdot \vec{b})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}\} + \{(\vec{b} \cdot \vec{c})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}\}}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$= 0$$

$$(ii) \vec{a} \cdot \vec{a}' + \vec{b} \cdot \vec{b}' + \vec{c} \cdot \vec{c}'$$

$$= \frac{\vec{a} \cdot \vec{b} \times \vec{c} + \vec{b} \cdot \vec{c} \times \vec{a} + \vec{c} \cdot (\vec{a} \times \vec{b})}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$= \frac{[\vec{a} \ \vec{b}' \ \vec{c}] + [\vec{a}' \ \vec{b} \ \vec{c}] + [\vec{a} \ \vec{b} \ \vec{c}']}{[\vec{a} \ \vec{b} \ \vec{c}]}$$

$$= 3$$

$$\begin{aligned}
 \textcircled{\text{iii}} \quad [\vec{a}', \vec{b}', \vec{c}'] &= \vec{a}' \cdot (\vec{b}' \times \vec{c}') \\
 &= \vec{a}' \cdot \frac{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})}{[\vec{a} \vec{b} \vec{c}]^2} \\
 &= \frac{\vec{a}'}{[\vec{a} \vec{b} \vec{c}]} \cdot \{ [\vec{c} \vec{a} \vec{b}] \vec{a} - [\vec{c} \vec{a} \vec{a}] \vec{b} \} \\
 &= \frac{\vec{a}'}{[\vec{a} \vec{b} \vec{c}]^2} \cdot [\vec{c} \vec{a} \vec{b}] \vec{a} \\
 &= \frac{\vec{a}' \cdot \vec{a}}{[\vec{a} \vec{b} \vec{c}]} \\
 &= \frac{\vec{b} \times \vec{c} \cdot \vec{a}}{[\vec{a} \vec{b} \vec{c}]^2} \\
 &= \frac{1}{[\vec{a} \vec{b} \vec{c}]}
 \end{aligned}$$

$$\therefore [\vec{a} \vec{b} \vec{c}] [\vec{a}' \vec{b}' \vec{c}'] = 1$$

iv) we have

$$\begin{aligned}
 \frac{\vec{b}' \times \vec{c}'}{[\vec{a}' \vec{b}' \vec{c}']} &= \frac{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})}{[\vec{a} \vec{b} \vec{c}]^2 [\vec{a}' \vec{b}' \vec{c}']} \\
 &= \frac{[\vec{c} \vec{a} \vec{b}] \vec{a} - [\vec{c} \vec{a} \vec{a}] \vec{b}}{[\vec{a} \vec{b} \vec{c}]} \text{ by } \textcircled{\text{iii}} \\
 &= \frac{[\vec{a} \vec{b} \vec{c}] \vec{a}}{[\vec{a} \vec{b} \vec{c}]} \\
 &= \vec{a}
 \end{aligned}$$

(4)

similarly $\vec{b} = \frac{\vec{c}' \times \vec{a}'}{[\vec{a}' \ \vec{b}' \ \vec{c}']}$

and $\vec{c} = \frac{\vec{a}' \times \vec{b}'}{[\vec{a}' \ \vec{b}' \ \vec{c}']}$

Hence the result.

(v) (i) Since \vec{a} , \vec{b} , \vec{c} are non-coplanar any vector \vec{r} can be expressed as a linear combination of \vec{a} , \vec{b} , \vec{c} .

Suppose,

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} \quad \text{--- (1)}$$

Taking dot with \vec{a}' we get,

$$\begin{aligned} \vec{r} \cdot \vec{a}' &= x\vec{a} \cdot \vec{a}' + y\vec{b} \cdot \vec{a}' + z\vec{c} \cdot \vec{a}' \\ &= x \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]} + y \frac{\vec{b} \cdot \vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]} \\ &\quad + z \frac{\vec{c} \cdot \vec{b} \times \vec{c}}{[\vec{a} \ \vec{b} \ \vec{c}]} \end{aligned}$$

$$= x$$

similarly $y = \vec{r} \cdot \vec{b}'$

and $z = \vec{r} \cdot \vec{c}'$

\therefore (1) becomes

$$\vec{r} = \vec{r} \cdot \vec{a}' \vec{a} + \vec{r} \cdot \vec{b}' \vec{b} + \vec{r} \cdot \vec{c}' \vec{c}$$

(II) Since vector \vec{a}' , \vec{b}' , \vec{c}' are non-coplanar any linear combination can be expressed as a combination of \vec{a}' , \vec{b}' , \vec{c}' . (5)

Suppose

$$\vec{x} = \lambda \vec{a}' + \mu \vec{b}' + \nu \vec{c}' \quad \text{--- (2)}$$

Taking dot with \vec{a} , we get

$$\begin{aligned} \vec{x} \cdot \vec{a} &= \lambda \vec{a}' \cdot \vec{a} + \mu \vec{b}' \cdot \vec{a} + \nu \vec{c}' \cdot \vec{a} \\ &= \lambda \frac{\vec{b}' \times \vec{c}' \cdot \vec{a}}{[\vec{a}' \vec{b}' \vec{c}']} + \mu \frac{\vec{c}' \times \vec{a}' \cdot \vec{a}}{[\vec{a}' \vec{b}' \vec{c}']} \\ &\quad + \nu \frac{\vec{a}' \times \vec{b}' \cdot \vec{a}}{[\vec{a}' \vec{b}' \vec{c}']} \end{aligned}$$

Similarly $\mu = \frac{\vec{x} \cdot \vec{b}}{\vec{a}' \cdot \vec{b}'}$
and $\nu = \frac{\vec{x} \cdot \vec{c}}{\vec{a}' \cdot \vec{c}'}$

\therefore (2) becomes

$$\vec{x} = \frac{\vec{x} \cdot \vec{a}}{\vec{a}' \cdot \vec{a}'} \vec{a}' + \frac{\vec{x} \cdot \vec{b}}{\vec{a}' \cdot \vec{b}'} \vec{b}' + \frac{\vec{x} \cdot \vec{c}}{\vec{a}' \cdot \vec{c}'} \vec{c}'$$

Suppose,

$$\vec{x} = l \vec{i} + m \vec{j} + n \vec{k} \quad \text{--- (3)}$$

Taking dot \vec{i} we get

$$\begin{aligned} \vec{x} \cdot \vec{i} &= l \vec{i} \cdot \vec{i} + m \vec{j} \cdot \vec{i} + n \vec{k} \cdot \vec{i} \\ &= l \end{aligned}$$

Similarly $m = \vec{x} \cdot \vec{j}$
and $n = \vec{x} \cdot \vec{k}$

\therefore (3) becomes $\vec{x} = \vec{x} \cdot \vec{i} \vec{i} + \vec{x} \cdot \vec{j} \vec{j} + \vec{x} \cdot \vec{k} \vec{k}$