Vector algebra – Problems (1)

(x) The points $4\vec{i}+5\vec{j}+\vec{k}$, $-\vec{j}-\vec{k}$, $3\vec{i}+9\vec{i}+4\vec{k}$ and $-4\vec{i}+4\vec{j}+4\vec{k}$ are coplaner.

Ans. Let the given points be denoted respectively by A, B, C, D. Then

$$\vec{BA} = 4\vec{i}+6\vec{j}+2\vec{k}$$
$$\vec{BC} = 3\vec{i}+10\vec{j}+5\vec{k}$$
$$\vec{BD} = -4\vec{i}+5\vec{j}+5\vec{k}$$

$$\therefore [\vec{BA}\ \vec{BC}\ \vec{BD}] = \begin{vmatrix} 4 & 6 & 2 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix}$$

$$= 4(50-25) - 6(15+20) + 2(15+40)$$

$$= 100 - 210 + 110$$

$$= 0$$

$\therefore$ The vectors $\vec{BA}$, $\vec{BC}$, $\vec{BD}$ are coplaner.

$\therefore$ The points B, A, C, D are coplaner.

(xi) $\vec{d}\cdot[\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}] = (\vec{b}\cdot\vec{d})[\vec{a}\ \vec{c}\ \vec{d}]$

$$\text{L.H.S.} = \vec{d}\cdot\vec{a}\times\{\vec{b}\cdot\vec{d}\,\vec{c} - \vec{b}\cdot\vec{c}\,\vec{d}\}$$
$$= \vec{d}\cdot\{(\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d})\}$$
$$= (\vec{b}\cdot\vec{d})[\vec{d}\ \vec{a}\ \vec{c}] - (\vec{b}\cdot\vec{c})[\vec{d}\ \vec{a}\ \vec{d}]$$
$$= (\vec{b}\cdot\vec{d})[\vec{a}\ \vec{c}\ \vec{d}] - 0$$
$$= \text{R.H.S.}$$

(xii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})+(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})=0$

$$\text{L.H.S.}=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}+\begin{vmatrix}\vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{d}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{d}\end{vmatrix}+\begin{vmatrix}\vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{d}\\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{d}\end{vmatrix}$$

$$=\{(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})\}$$
$$+\{(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})-(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})\}$$
$$+\{(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})-(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})\}$$
$$=0$$
$$=\text{R.H.S.}$$

(xiii) If $\vec{a},\vec{b},\vec{c}$ are coplaner then $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ are also non-coplaner

Ans. since $\vec{a},\vec{b},\vec{c}$ are non-coplaner

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}]\neq 0 \quad \text{— (1)}$$

Now, $[\vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\quad \vec{a}\times\vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$ from (vii)

$\therefore \vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ $\neq 0$ by (1) are non coplaner

Note:- Conversely if $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ are non-coplanar, then $\vec{a},\vec{b},\vec{c}$ are also non-coplanar.