Q.3) # VECTOR - ALGEBRA

Show that
(i) dot in a scalar triple product and cross can be interchanged.
(ii) vectors can be interchanged cyclically without changing the value of the triple product.
(iii) an interchange of vectors non-cyclically changes the sign of the triple - product.
(iv) If the vectors are equal or collinear (parallel) the triple product is zero.

Proof:- with unit vectors $\vec{i}, \vec{j}, \vec{k}$ along the rectangular cartesian coordinate axis
let,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$
$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$
$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

(i) Then $\vec{a}\cdot\vec{b}\times\vec{c} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{——— (1)}$$

similarly

$$\vec{a} \times \vec{b} \cdot \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ ———}$$

From ① & ②

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}$$

This proves (i)

(ii) As in (i), we have

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ ——— } ①$$

similarly

$$\vec{b} \cdot \vec{c} \times \vec{a} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ ——— } ③$$

and $\vec{c} \cdot \vec{a} \times \vec{b} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ — (4) (3)

From (1), (3) and (4) we get

$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{b} \cdot \vec{c} \times \vec{a} = \vec{c} \cdot \vec{a} \times \vec{b}$

This proves (ii)

(iii) $\vec{a} \cdot \vec{b} \times \vec{c} = -\vec{a} \cdot \vec{c} \times \vec{b}$

Denoting $\vec{a} \cdot \vec{b} \times \vec{c}$ (which is the same as $\vec{a} \times \vec{b} \cdot \vec{c}$) by $[\vec{a}\ \vec{b}\ \vec{c}]$ — (5)

shows that

$[a\ b\ c] = -[a\ c\ b]$

This proves (iii)

(iv) Suppose $\vec{a}$ and $\vec{b}$ are collinear (or parallel)

then $\vec{b} = k\vec{a}$, where k is a scalar.

$\therefore [a\ b\ c] = [\vec{a}\ \ k\vec{a}\ \ \vec{c}]$

$= \vec{a} \times k\vec{a} \cdot \vec{c}$

$= k\vec{a} \times \vec{a} \cdot \vec{c}$

$= 0$

This proves (iv)