Vector algebra - Problems

(xiv) $[\vec{l}\ \vec{m}\ \vec{n}]\,[\vec{p}\ \vec{q}\ \vec{r}] = \begin{vmatrix} \vec{l}\cdot\vec{p} & \vec{l}\cdot\vec{q} & \vec{l}\cdot\vec{r} \\ \vec{m}\cdot\vec{p} & \vec{m}\cdot\vec{q} & \vec{m}\cdot\vec{r} \\ \vec{n}\cdot\vec{p} & \vec{n}\cdot\vec{q} & \vec{n}\cdot\vec{r} \end{vmatrix}$

Ans. Let $\vec{\alpha}$ be any vector. Then expanding $(\vec{l}\times\vec{m})\times(\vec{n}\times\vec{\alpha})$ in two ways we get

$$[\vec{l}\ \vec{m}\ \vec{\alpha}]\vec{n} - [\vec{l}\ \vec{m}\ \vec{n}]\vec{\alpha} = [\vec{l}\ \vec{n}\ \vec{\alpha}]\vec{m} - [\vec{m}\ \vec{n}\ \vec{\alpha}]\vec{l}$$

or, $$[\vec{l}\ \vec{m}\ \vec{n}]\vec{\alpha} = [\vec{m}\ \vec{n}\ \vec{\alpha}]\vec{l} - [\vec{l}\ \vec{n}\ \vec{\alpha}]\vec{m} + [\vec{l}\ \vec{m}\ \vec{\alpha}]\vec{n}$$

putting $\vec{\alpha} = \vec{p}\times\vec{q}$ this becomes

$$[\vec{l}\ \vec{m}\ \vec{n}]\,\vec{p}\times\vec{q} = [\vec{m}\ \vec{n}\ \vec{p}\times\vec{q}]\vec{l} - [\vec{l}\ \vec{n}\ \vec{p}\times\vec{q}]\vec{m} + [\vec{l}\ \vec{m}\ \vec{p}\times\vec{q}]\vec{n}$$

$$= (\vec{m}\times\vec{n})\cdot(\vec{p}\times\vec{q})\,l - (\vec{l}\times\vec{n})(\vec{p}\times\vec{q})\vec{m} + (\vec{l}\times\vec{m})\cdot(\vec{p}\times\vec{q})\vec{n}$$

Taking dot product with $\vec{r}$ we get

$$[\vec{l}\ \vec{m}\ \vec{n}]\,[\vec{p}\ \vec{q}\ \vec{r}] = \begin{vmatrix} \vec{m}\cdot\vec{p} & \vec{m}\cdot\vec{q} \\ \vec{n}\cdot\vec{p} & \vec{n}\cdot\vec{q} \end{vmatrix} \vec{l}\cdot\vec{r} - \begin{vmatrix} \vec{l}\cdot\vec{p} & \vec{l}\cdot\vec{q} \\ \vec{n}\cdot\vec{p} & \vec{n}\cdot\vec{q} \end{vmatrix} \vec{m}\cdot\vec{r} + \begin{vmatrix} \vec{l}\cdot\vec{p} & \vec{l}\cdot\vec{q} \\ \vec{m}\cdot\vec{p} & \vec{m}\cdot\vec{q} \end{vmatrix} \vec{n}\cdot\vec{r}$$

$$= \begin{vmatrix} \vec{l}\cdot\vec{p} & \vec{l}\cdot\vec{q} & \vec{l}\cdot\vec{r} \\ \vec{m}\cdot\vec{p} & \vec{m}\cdot\vec{q} & \vec{m}\cdot\vec{r} \\ \vec{n}\cdot\vec{p} & \vec{n}\cdot\vec{q} & \vec{n}\cdot\vec{r} \end{vmatrix}$$

(xv) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

Ans. Let $\vec{d}$ be any vector. Then expanding $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ in two ways we get

$$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

or, $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$

Putting $\vec{d} = \vec{a}\times\vec{b}$ this becomes

$$[\vec{a}\ \vec{b}\ \vec{c}]\ \vec{a}\times\vec{b} = (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{b})\vec{a} - (\vec{a}\times\vec{c})\cdot(\vec{a}\times\vec{b})\vec{b} + (\vec{a}\times\vec{b})(\vec{a}\times\vec{b})\vec{c}$$

Taking dot product with $\vec{c}$ we get

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \vec{a}\cdot\vec{c}\begin{vmatrix} \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} \end{vmatrix} - (\vec{b}\cdot\vec{c})\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} \end{vmatrix} + (\vec{c}\cdot\vec{c})\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

(xvi) $$\vec{c} = \frac{\begin{vmatrix} \vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{c}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{c}\cdot\vec{a} \\ \vec{a}\cdot\vec{b} & \vec{c}\cdot\vec{a} \end{vmatrix}\vec{b}}{\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}}$$

If $\vec{a}, \vec{b}, \vec{c}$ are coplaner and $\vec{a}$ is not parallel Collinear to $\vec{b}$.

Ans. Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar and $\vec{a}$ is not parallel to $\vec{b}$,

$\therefore \vec{c}$ can be expressed as a linear Combination of $\vec{a}$ and $\vec{b}$. Let therefore,

$$\vec{c} = x\vec{a} + y\vec{b} \quad \text{— (1)}$$

Taking cross product with $\vec{b}$, we get

$$\vec{c}\times\vec{b} = x\,\vec{a}\times\vec{b}$$

Taking dot product with $\vec{a}\times\vec{b}$ we get

$$(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b}) = x\,(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$

$$\text{or, } x = \frac{(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})}$$

Again, taking cross product of (1) with $\vec{a}$ we get

$$\vec{a}\times\vec{c} = y\,\vec{a}\times\vec{b}$$

Taking dot product with $\vec{a}\times\vec{b}$, we get

$$(\vec{a}\times\vec{c})\cdot(\vec{a}\times\vec{b}) = y\,(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$

$$\text{or, } y = \frac{(\vec{a}\times\vec{c})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})}$$