Vector algebra — Problems

(xvii) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar (linearly dependent) then $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also coplanar (linearly dependent).

Ans:- If $\vec{a}, \vec{b}, \vec{c}$ are coplanar (linearly dependent) then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ — (1)

Now, $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$ from (ii)

$= 0$ by (1)

$\therefore \vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also coplanar (linearly dependent).

(xviii) If $\vec{A} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $\vec{B} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $\vec{C} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ then

$$\vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} (\vec{a}\cdot\vec{b}\times\vec{c})$$

Answer we have $\vec{B}\times\vec{C} = (x_2y_2 - x_3y_2)\,\vec{a}\times\vec{b}$
$+ (z_2x_3 - z_3x_2)\,\vec{c}\times\vec{a}$
$+ (y_2x_3 - y_3z_2)\,\vec{b}\times\vec{c}$

$\therefore \vec{A}\cdot\vec{B}\times\vec{C} = x_1(x_2y_2 - x_3y_2)[\vec{a}\ \vec{a}\ \vec{b}]$
$+ x_1(y_2z_3 - y_3z_2)[\vec{a}\ \vec{b}\ \vec{c}]$
$+ x_1(z_2x_3 - z_3x_2)[\vec{a}\ \vec{b}\ \vec{c}]$ etc.

$$= x_1 (y_2 z_2 - y_3 z_2) [\vec{a}\ \vec{b}\ \vec{c}]$$
$$+ y_1 (x_2 z_3 - x_3 z_2) [\vec{a}\ \vec{b}\ \vec{c}]$$
$$+ z (x_2 y_3 - x_3 y_2) [\vec{a}\ \vec{b}\ \vec{c}]$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

(xix) $[\vec{l}\ \vec{m}\ \vec{n}]\ \vec{a} \times \vec{b} = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$

if $\vec{l}, \vec{m}, \vec{n}$ are non - coplanar.

Ans. Let $\vec{d}$ be any vector. Then expanding $(\vec{l} \times \vec{m}) \times (\vec{n} \times \vec{d})$ in two ways we have

$$[\vec{l}\ \vec{m}\ \vec{d}]\vec{n} - [\vec{l}\ \vec{m}\ \vec{n}]\vec{d}$$
$$= [\vec{l}\ \vec{n}\ \vec{d}]\vec{m} - [\vec{m}\ \vec{n}\ \vec{d}]\vec{l}$$

or $[\vec{l}\ \vec{m}\ \vec{n}]\vec{d} = [\vec{m}\ \vec{n}\ \vec{d}]\vec{l} - [\vec{l}\ \vec{n}\ \vec{d}]\vec{m} + [\vec{l}\ \vec{m}\ \vec{d}]\vec{n}$

putting $\vec{d} = \vec{a} \times \vec{b}$ this becomes

$$[\vec{l}\ \vec{m}\ \vec{n}]\ \vec{a} \times \vec{b} = (\vec{m} \times \vec{n})\cdot(\vec{a} \times \vec{b})\vec{l}$$
$$- (\vec{l} \times \vec{n})\cdot(\vec{a} \times \vec{b})\vec{m}$$
$$+ (\vec{l} \times \vec{m})\cdot(\vec{a} \times \vec{b})\vec{n}$$

$$= \begin{vmatrix} \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} \end{vmatrix}\vec{l} - \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} \end{vmatrix}\vec{m}$$

$$+ \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} \end{vmatrix}\vec{n}$$

$$= \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$$

<u>Note</u>:- $\vec{l}$, $\vec{m}$, $\vec{n}$ need not be non-coplanar.

(xx) If $\vec{a}$. $\vec{b}$, $\vec{c}$ are non-coplanar, then

$$\vec{d} = \frac{\vec{c}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{a}\times\vec{b} + \frac{\vec{a}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{b}\times\vec{c} + \frac{\vec{b}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}\vec{c}\times\vec{a}$$ for any vector $\vec{d}$.

<u>Ans.</u> Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are non coplanar we have $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ —— ①

Now, $[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]^2$ [from (ii)]

$\neq 0$ by ①

$\therefore$ $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ are also non-coplanar.

Any vector $\vec{d}$ can be expressed as a linear Combination of $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$.

let therefore

$$\vec{d} = x\,\vec{a}\times\vec{b} + y\,\vec{b}\times\vec{c} + z\,\vec{c}\cdot\vec{a} \quad \text{———— } (2)$$

Taking dot product with c we get

$$\vec{c}\cdot\vec{d} = x\,\vec{c}\cdot\vec{a}\times\vec{b} + y\,\vec{c}\cdot\vec{b}\times\vec{c} + z\,\vec{c}\cdot\vec{c}\times\vec{a}$$

$$= x\,[\vec{a}\;\vec{b}\;\vec{c}] + 0 + 0$$

$$\therefore \quad x = \frac{\vec{c}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

similarly $y = \dfrac{\vec{a}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

any $\quad z = \dfrac{\vec{b}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

Putting values of x, y, z in (2), we get

$$\vec{d} = \frac{\vec{c}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}\,\vec{a}\times\vec{b} + \frac{\vec{a}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}\,\vec{b}\times\vec{c} + \frac{\vec{b}\cdot\vec{d}}{[\vec{a}\;\vec{b}\;\vec{c}]}\,\vec{c}\times\vec{a}$$