7.) Reciprocal system of vectors.

(α) Define reciprocal system of vectors.

(β) If $\vec{a'}, \vec{b'}, \vec{c'}$ be reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$, prove that

(i) $\vec{a} \times \vec{a'} + b \times \vec{b'} + \vec{c} \times \vec{c'} = 0$

(ii) $\vec{a} \cdot \vec{a'} + \vec{b} \cdot \vec{b'} + \vec{c} \cdot \vec{c'} = 3$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a'}\ \vec{b'}\ \vec{c'}] = 1$

(iv) The set of vectors $\vec{a}, \vec{b}, \vec{c}$ is the reciprocal system of vectors $\vec{a'}, \vec{b'}, \vec{c'}$.

(v) Any vector $\vec{r}$ can be expressed as

(I) $\vec{r} = \vec{r} \cdot \vec{a'}\,\vec{a} + \vec{r} \cdot \vec{b'}\,\vec{b} + \vec{r} \cdot \vec{c'}\,\vec{c}$

(II) $\vec{r} = \vec{r} \cdot \vec{a}\,\vec{a'} + \vec{r} \cdot \vec{b}\,\vec{b'} + \vec{r} \cdot \vec{c}\,\vec{c'}$

(III) $\vec{r} = \vec{r} \cdot \vec{i}\,\vec{i} + \vec{r} \cdot \vec{j}\,\vec{j} + \vec{r} \cdot \vec{k}\,\vec{k}$

sol (α) Definition :- If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar the vectors $\vec{a'}, \vec{b'}, \vec{c'}$ defined by

$$\vec{a'} = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},$$

$$\vec{b'} = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\vec{c'} = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

are called reciprocal to the vectors $\vec{a}, \vec{b}, \vec{c}$.

(B) (i) we have,

$$\vec{a} \times \vec{a'} + \vec{b} \times \vec{b'} + \vec{c} \times \vec{c'}$$

$$= \frac{\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\} + \{(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}\} + \{(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\{(\vec{c}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\} + \{(\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}\} + \{(\vec{b}\cdot\vec{c})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= 0$$

(ii) $\vec{a} \cdot \vec{a'} + b \cdot \vec{b'} + \vec{c} \cdot \vec{c'}$

$$= \frac{\vec{a} \cdot \vec{b} \times \vec{c} + \vec{b} \cdot \vec{c} \times \vec{a} + \vec{c} \cdot (\vec{a} \times \vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= 3$$

(iii) $[\vec{a'}, \vec{b'}, \vec{c'}] = \vec{a'} \cdot \vec{b'} \times \vec{c'}$

$$= \vec{a'} \cdot \frac{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{\vec{a'}}{[\vec{a}\ \vec{b}\ \vec{c}]} \cdot \{[\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{a}\ \vec{a}]\vec{b}\}$$

$$= \frac{\vec{a'}}{[\vec{a}\ \vec{b}\ \vec{c}]^2} \cdot [\vec{c}\ \vec{a}\ \vec{b}]\vec{a}$$

$$= \frac{\vec{a'} \cdot \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{\vec{b} \times \vec{c} \cdot \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]^2}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}][\vec{a'}\ \vec{b'}\ \vec{c'}] = 1$$

(iv) we have

$$\frac{\vec{b'} \times \vec{c'}}{[\vec{a'}\ \vec{b'}\ \vec{c'}]} = \frac{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]^2 [\vec{a'}\ \vec{b'}\ \vec{c'}]}$$

$$= \frac{[\vec{c}\ \vec{a}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{a}\ \vec{a}]\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} \text{ by (iii)}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \vec{a}$$

similarly $\vec{b} = \dfrac{\vec{c}' \times \vec{a}'}{[\vec{a}' \; \vec{b}' \; \vec{c}']}$

and $\vec{c} = \dfrac{\vec{a}' \times \vec{b}'}{[\vec{a}' \; \vec{b}' \; \vec{c}']}$

Hence the result.

(v) (I) Since $\vec{a}$ $\vec{b}$ $\vec{c}$ are non - coplanar any vector $\vec{r}$ can be expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$.

suppose,

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} \quad \text{—— (1)}$$

Taking dot with $\vec{a}'$ we get,

$$\vec{r} \cdot \vec{a}' = x\vec{a} \cdot \vec{a}' + y\vec{b} \cdot \vec{a}' + z\vec{c} \cdot \vec{a}'$$

$$= x \frac{\vec{a} \cdot \vec{b} \times \vec{c}}{\vec{a} \; \vec{b} \; \vec{c}} + y \frac{\vec{b} \cdot \vec{b} \times \vec{c}}{\vec{a} \; \vec{b} \; \vec{c}} + z \frac{\vec{c} \cdot \vec{b} \times \vec{c}}{[\vec{a} \; \vec{b} \; \vec{c}]}$$

$$= x$$

similarly $y = \vec{r} \cdot \vec{b}$

and $z = \vec{r} \cdot \vec{c}'$

$\therefore$ (1) becomes

$$\vec{r} = \vec{r} \cdot \vec{a}' \, \vec{a} + \vec{r} \cdot \vec{b}' \, \vec{b} + \vec{r} \cdot \vec{c}' \, \vec{c}$$

(II) since $\vec{a}', \vec{b}', \vec{c}'$ are non-coplanar any vector $\vec{r}$ can be expressed as a linear combination of $\vec{a}', \vec{b}', \vec{c}'$.

Suppose

$$\vec{r} = \lambda \vec{a}' + \mu \vec{b}' + \nu \vec{c}' \quad \text{--- (2)}$$

Taking dot with $\vec{a}$, we get

$$\vec{r}\cdot\vec{a} = \lambda \vec{a}'\cdot\vec{a} + \mu \vec{b}'\cdot\vec{a} + \nu \vec{c}'\cdot\vec{a}$$

$$= \lambda \frac{\vec{b}\times\vec{c}\cdot\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \mu \frac{\vec{c}\times\vec{a}\cdot\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \nu \frac{\vec{a}\times\vec{b}\cdot\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \lambda$$

Similarly $\mu = \vec{r}\cdot\vec{b}$

and $\nu = \vec{r}\cdot\vec{c}$

$\therefore$ (2) becomes

$$\vec{r} = \vec{r}\cdot\vec{a}\,\vec{a}' + \vec{r}\cdot\vec{b}\,\vec{b}' + \vec{r}\cdot\vec{c}\,\vec{c}'$$

(III) Suppose,

$$\vec{r} = l\vec{i} + m\vec{j} + n\vec{k} \quad \text{--- (3)}$$

Taking dot $\vec{i}$ we get

$$\vec{r}\cdot\vec{i} = l\vec{i}\cdot\vec{i} + m\vec{j}\cdot\vec{i} + n\vec{k}\cdot\vec{i}$$

$$= l$$

Similarly $m = \vec{r}\cdot\vec{j}$

and $n = \vec{r}\cdot\vec{k}$

$\therefore$ (3) becomes $\vec{r} = \vec{r}\cdot\vec{i}\,\vec{i} + \vec{r}\cdot\vec{j}\,\vec{j} + \vec{r}\cdot\vec{k}\,\vec{k}$