Abstract Algebra

PRIME AND COMPOSITE ELEMENTS OF INTEGRAL DOMAIN :-

Definition (prime):- Let D be an integral domain. Then a non-unit element of D is called prime, if it has no divisors other than its unit or associates.

Definition (Composite) :- An element in D which is not prime is called composite. Thus, if an element a of D is prime and $a = a_1 a_2$ where $a_1, a_2 \in D$. then one of a_1 and a_2 must be a unit in D.

Definition (Co-prime element) Any two non-zero elements of an integral domain are said to be co-prime or relatively prime if their greatest common divisor is unity.

Unique factorisation domain

Definition:- An integral domain D is said to be a unique factorization domain if every non-zero element of D is either a unit or it is expressible as the product of a finite number of prime elements of D and this factorisation apart from order and associates is unique.

Notes:-

(i) A principal ideal domain is a unique factorisation domain.

Theorem :- Any two elements of a unique factorisation domain possesses greatest common divisor as well as lowest common multiple.

Proof :- Let D be a unique factorization domain and let a and b be any two elements of D.

Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ —— (1)

and $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$ —— (2)

Here we have arranged both the expression (1) and (2) in such a way that the same prime factors appear in both by employing the integer zero as index in any case, if necessary.

Since p_1, p_2, p_r are all distinct primes and $\alpha_1, \alpha_2, \ldots \alpha_r$; $\beta_1, \beta_2, \ldots \beta_r$ are all non-negative integers.

Let $d_i = \max\{\alpha_i, \beta_i\}$ and $l_i = \min\{\alpha_i, \beta_i\}$ then $p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r}$ and $p_1^{d_1} p_2^{d_2} \cdots p_r^{d_r}$ are clearly the greatest common divisor and the lowest common multiple of a and b respectively.