Abstract Algebra

Theorem :- If H and K are subgroups of an abelian group G, then HK is a subgroup of G.

Proof :- Let us suppose H and K are subgroups of an abelian group G, so that

$$H^{-1} = H, \quad K^{-1} = K \quad \text{—— (1)}$$

$$\text{and } HH^{-1} = H, \quad KK^{-1} = K \quad \text{—— (2)}$$

To show that (HK) is a subgroup of G, we have to show that

$$(HK)(HK)^{-1} = HK$$

Now, since G is abelian $\Rightarrow ab = ba$
$\forall\, a, b \in G$

$$\therefore h \in H,\ k \in K \Rightarrow hk = kh \Rightarrow HK = KH \quad \text{—— (3)}$$

Consider $(HK)(HK)^{-1} = (HK)(K^{-1}H^{-1})$

$= H(KK^{-1})H^{-1}$

$= HKH^{-1} = (HK)H^{-1}$

$= (KH)H^{-1}$

$= K(HH^{-1})$

$= KH = HK$

UNION AND INTERSECTION OF SUBGROUPS

Theorem :- The intersection of any two subgroups of a group G is a subgroup of G.

Proof :- Let H_1 and H_2 be two subgroups of a group G. To show that $H_1 \cap H_2$ is a subgroup of G. For this we have to show that $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Let $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$

Now, since H_1 and H_2 are subgroups, then

$$a, b \in H_1 \Rightarrow ab^{-1} \in H_1$$

$$\text{and } a, b \in H_2 \Rightarrow ab^{-1} \in H_2$$

$$ab^{-1} \in H_1 \text{ and } ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$$

Remarks

- $H_1 \cap H_2$ is a largest subgroup of G which is contained in H_1 as well as H_2

Theorem :- The union of two subgroups of a group G is a subgroup of G iff one is contained in the other.

Proof :- Let H_1 and H_2 be subgroups of a group G.

Let $H_1 \subset H_2$ or $H_2 \subset H_1$.

To show that $H_1 \cup H_2$ is a subgroup of G.

$$H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$$

Also, H_2 is a subgroup of G
$\Rightarrow H_1 \cup H_2$ is a subgroup of G. Again
$H_2 \subset H_1 \Rightarrow H_1 \cup H_2 = H_1$
Also, H_1 is a subgroup of G
$\Rightarrow H_1 \cup H_2$ is a subgroup of G.
Hence, $H_1 \cup H_2$ is a subgroup of G, in both cases.

Conversely, suppose that H_1 and H_2 are are subgroups of a group G such that $H_1 \cup H_2$ is a subgroup of G.
To show

$$H_1 \subset H_2 \text{ or } H_2 \subset H_1$$

We suppose the contrary.
Then $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$

$H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1$ s.t. $a \notin H_2$
$H_2 \not\subset H_1 \Rightarrow \exists\, b \in H_2$ s.t. $b \notin H_1$

and
Now $a, b \in H_1 \cup H_2$, and $H_1 \cup H_2$ is a subgroup of G.

$$\Rightarrow ab \in H_1 \cup H_2$$

This implies $ab \in H_1$ or $ab \in H_2$

$a \in H_1,\ ab \in H_1 \Rightarrow a^{-1}(ab) \in H_1$
$\Rightarrow (a^{-1}a)b \in H_1 \Rightarrow eb \in H_1 \Rightarrow b \in H_1$
which is a contradiction

$b \in H_2,\ ab \in H_2 \Rightarrow (ab)b^{-1} \in H_2$
$\Rightarrow a \in H_2$ (For $(ab)b^{-1} = a(bb^{-1}) = ae = a$)

Again we get a Contradiction

Hence our initial assumption is wrong.

Consequently $H_1 \subset H_2$ or $H_2 \subset H_1$

Remark

- The union of two subgroups is not necessarily a subgroup.

Example :- Let H_1 and H_2 be two subgroups of the group $(Z, +)$ where

$H_1 = \{2n : n \in Z\}$, $H_2 = \{5n : n \in Z\}$

Then $H_1 \cup H_2 = \{x : x = 2n \text{ or } 5n \text{ where } n \in Z\}$

$= \{0, \pm 2, \pm 4, \pm 6, \ldots 0, \pm 5, \pm 10, \pm 15 \ldots\}$

$6, 15 \in H_1 \cup H_2 \Rightarrow 6 + 15 = 21 \notin H_1 \cup H_2$

$\Rightarrow$ closure property is not satisfied

Hence, $H_1 \cup H_2$ is not a subgroup of $(Z, +)$.