Uniform Convergence of sequences and Series of Functions :-

Sequence in a set

Definition :- A sequence is a set X is a mapping of the set N of positive integers into X.

The image $s(n)$ of $n \in N$ is usually denoted by s_n. We call s_n the n^{th} term of the sequence. The sequence s is often written as $\{s_1, s_2, \dots, s_n, \dots\}$ or simply as $\{s_n\}$.

Convergence of sequences in a metric space :-

Definition :- A sequence $\{s_n\}$ in a metric space (X, d) is said to Converge to a point $x_0 \in X$ iff for every $\epsilon > 0$, there exists a positive integer $m(\epsilon)$ such that

$$n \geq m(\epsilon) \implies d(x_0, s_n) < \epsilon$$

or, equivalently, for each open sphere $S(x_0, \epsilon)$ with Centre x_0 and arbitrary radius ϵ there exists a positive integer $m(\epsilon)$ such that

$$n \geq m(\epsilon) \Rightarrow s_n \in S(x_0, \epsilon).$$

In symbols, we express this fact by

$$s_n \to x_0 \text{ or } \lim s_n = x_0 .$$

Cauchy's sequence in a Metric space

Definition :- Let (X, d) be a metric space and $S = \{S_n\}$ be a sequence in X. Then S is said to be a Cauchy sequence in X iff for every $\epsilon > 0$, there exists a positive integer $m(\epsilon)$ such that

$$p \geq m(\epsilon),\ n \geq m(\epsilon) \Rightarrow d(S_p, S_n) < \epsilon.$$

It is easy to see that every convergent sequence in a metric space is a Cauchy sequence but the converse is not true in general.

Uniformly bounded sequences :-

Definition :- A sequence $\{f_n\}$ of real valued functions defined on a metric space X is said to be uniformly bounded on X if

$$|f_n(x)| < M \text{ for every } x \in X.$$

and for every positive integer n.

Examples :- (i) The sequence $\{f_n\}$ where $f_n(x) = 1/nx$ is bounded but not uniformly bounded, on X if

$$|f_n(x)| \leq M \text{ for every } x \in X.$$

and for every positive integer n.

(ii) since $|\sin nx| \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the sequence defined by $\{\sin nx\}$ is uniformly bounded on $\mathbb{R}$.

Uniform Convergence :-

We shall consider sequence of real valued functions defined on a metric space (X, d).

Limit function of a sequence of functions.

Let f_n be a real valued function defined on a metric space (X, d) for all positive integral values n. We suppose that the function f_n tends to a definite limit for all values of x in X as $n \to \infty$. This means that if c is any point in X, the sequence of constant terms.

$\{f_n(c)\}$, i.e. $f_1(c), f_2(c), \ldots, f_n(c) \ldots$

is convergent. The limiting values of such sequences for various point in X will define a function of x, say f, such that the limiting value of the convergent sequence $\{f_n(c)\}$ is equal to the value $f(c)$ of the function f at c. This function f is called the limit function or the limit of the convergent sequence $\{f_n\}$.