Uniform Convergence of Sequence and series of functions

Point-wise convergence and uniform Convergence

Point-wise Convergence :-

Definition: Let (f_n), $n = 1, 2, 3, \ldots$ be a sequence of real-valued functions defined on a non empty set X i.e. $f_n : X \to R$ for each $n \in N$. To each point $c \in X$ there Corresponds a sequence $(f_n(c))$ of real terms

$$f_1(c), f_2(c), f_3(c), \ldots\ldots$$

We suppose that the sequence $(f_n(c))$ of real terms Converges for every $c \in X$.

Let $(f_n(c))$ Converges to $f(c)$.

In this way let the sequences at all points $c, d, e, \ldots$ of X Converges to $f(c), f(d), f(e), \ldots$

Thus we define in a natural way, a real-valued function f, with domain X and range the set defined by 1, so that its value $f(d)$ for $d \in X$ is $\lim (f_n(d))$..

$$\text{Thus } f(x) = \lim_{n \to \infty} (f_n(x)), \ \forall x \in X \quad \text{—} (2)$$

The function f, thus defined, is known as the limit or the point-wise limit of the sequence (f_n) on X, and the sequence (f_n) is said to be <u>pointwise Convergent</u> to f on X.

<u>Definition</u> :- If the series $\sum f_n$ (of real valued function f_n defined on X) Converges for every point $x \in X$, and we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \forall\ x \in X \quad \text{—— (3)}$$

the function f is called the sum or the point-wise sum of the series $\sum f_n$ on X.

Thus if a function f is the pointwise limit of pointwise Convergent sequence (f_n) of functions defined on X, then to each $\epsilon > 0$ and to each $x \in X$, there Corresponds a positive integer m such that

$$|f_n(x) - f(x)| < \epsilon, \forall\ n \geq m \quad \text{—— (4)}$$

If we fix ϵ, the choice of m may depend upon the choice of x.

<u>Uniform Convergence</u> :- A sequence (f_n) of real valued function with domain X is said to be uniformly Convergent on X to a real-valued function f defined on X if for any $\epsilon > 0$ and for all $x \in X$ there Corresponds a positive integer m (independent of x but dependent of ϵ) such that for all $x \in X$,

$$|f_n(x) - f(x)| < \epsilon, \forall\ n \geq m \quad \text{—— (5)}$$

Also in this situation we say that the function f is the uniform limit of the sequence (f_n).