Example of Pointwise Convergence It is clear that every uniformly Convergent sequence is pointwise Convergent, and the uniform limit function is the same as the point-wise limit function.

Q.) Construct an example of a point-wise Convergent sequence which is not uniformly Convergent.)

Ans:- We Consider the sequence (f_n) of real valued functions defined on the real line $\mathbb{R}$ by

$$f_n(x) = \frac{nx}{1+n^2x^2} \text{ for all } x \in \mathbb{R}.$$

For each fixed $x \in \mathbb{R}$

$$f_n(x) = \frac{x/n}{1/n^2 + x^2} \to 0 \quad \text{as } n \to \infty$$

Thus the sequence (f_n) is pointwise Convergent with the function f defined by $f(x)=0$ for all $x \in \mathbb{R}$, as the point-wise limit.

We show that the sequence (f_n) is not uniformly Convergent in any interval $[a, b]$ on $\mathbb{R}$ with, 0 as an interior point.

Suppose that (f_n) is uniformly Convergent in any $[a, b]$ so that the pointwise limit f is also the uniform limit.

Let $\epsilon > 0$ be given. Then there exists a positive integer m such that $\forall x \in [a,b]$ and $\forall\ n \geq m$

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \epsilon$$

If we take $\epsilon = \frac{1}{4}$ and k an integer $> m$ such that $\frac{1}{k} \in [a,b]$

we find on taking $n = k$ and $x = \frac{1}{k}$ that

$$\frac{nx}{1+n^2x^2} = \frac{k \cdot \frac{1}{k}}{1+k^2 \cdot \frac{1}{k^2}} = \frac{1}{1+1} = \frac{1}{2} \nless \frac{1}{4}$$

Thus we arrive at a contradiction. Therefore, the sequence is not uniformly convergent in the interval $[a,b]$, which contains the point $1/k$.

But since $1/k \to 0$, the interval $[a,b]$ contains 0.

Hence the sequence (f_n) is not uniformly convergent in any interval $[a,b]$ containing 0 even though it is pointwise convergent there.

Q.) Show that the sequence (f_n), where

$$f_n(x) = \frac{1}{x+n}$$

is uniformly convergent in any interval $[0,b]$, $b > 0$

Sol:- For each fixed $x \in [0,b]$,

$$f_n(x) = \frac{1/n}{\frac{x}{n}+1} \to 0 \text{ as } n \to \infty$$

Hence the sequence (f_n) convergence pointwise to the function f defined by $f(x) = 0$ for all $x \in [0,b]$, For any $\epsilon > 0$

$$|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| = \frac{1}{x+n} < \epsilon$$

If $n > (1/\epsilon) - x$. But $(1/\epsilon) - x$ decreases as x increases and its maximum values is $1/\epsilon$ at $x = 0$. Let m be a positive integer $\geq 1/\epsilon$ so that for $\epsilon > 0$, there exists m such that

$$|f_n(x) - f(x)| < \epsilon, \forall\ n \geq m$$

Hence the sequence (f_n) is uniformly convergent in any interval $[0, b]$ with $b > 0$.