SOME DEFINITIONS :-

BASIC SOLUTION (B.S) :-

We Consider a system $Ax = b$ of m equations in n unknowns (variables, $n > m$) and $r(A) = r(Ab) = m$.

If any $m \times n$ non-singular matrix (whose determinant is not zero) is chosen from A and if all the $(n-m)$ variables not associated with the columns of this matrix are set equal to zero, the solution to the resulting system of equation is called a basic solution. In other words a solution obtained by setting any $(n-m)$ variables to zero is called a basic solution, provided the determinant of the coefficients of the remaining m variables is not zero. Thus in a basic solution at least $n-m$ variables must vanish.

The m variables associated with the columns of the above non-singular matrix which may be different from zero are called basic variables.

It is important to note that the matrix formed by the coefficients of m basic variables are L.I. Thus a solution in which the vectors associated to m variables are L.I. and the remaining $n-m$ variables are zero is called a basic solution.

LINEAR PROGRAMMING

If B is the matrix of m L.I. vectors of A and x_B is the vector of the corresponding variables (basic variables), then the basic solution is given by:

$$x_B = B^{-1} b$$

Since m vectors out of n (number of columns of coefficient matrix A) can be selected in ${}^nC_m = \frac{n!}{m!\,n-m!}$ ways.

Hence maximum number of basic solution is

$${}^nC_m = \frac{n!}{m!\,n-m!}$$

Basic solutions are of two types.

1. Non-degenerate B.S. :- A B.S. is called non-degenerate B.S. if non of the basic variables is zero. In other words all the m basic variables are non zero.
2. Degenerate B.S. :- A B.S. is called degenerate B.S. if one or more of the basic variables are zero.

Theorem :- A necessary and sufficient condition for the existence and non-degeneracy of all the basic solutions of $Ax = b$ is that every set of m columns of the augmented matrix $Ab = [A, b]$ is L.I.

Proof :- Necessary Condition

First we consider that all the basic solutions of $Ax = b$ exist and are non-degenerate. Therefore every set of m- Columns vectors of A are L.I. . Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be one set of m- Columns vectors of A, then from the given system we have

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = b.$$

But $x_1 \neq 0$. since each solution is non-degenerate. Therefore the vector b can replace α_1 in the basis $\alpha_1, \alpha_2, \ldots, \alpha_m$ By replacement theorem of vectors). Thus the vectors $b, \alpha_2, \alpha_3, \ldots, \alpha_m$ also form a basis and hence are L.I.. In the similar way $\alpha_1, b, \alpha_3, \ldots, \alpha_m$; $\alpha_1, \alpha_2, b, \alpha_4, \ldots, \alpha_m$ etc are L.I. .

Thus the vector b with $(m-1)$ vectors of A form a L.I. set.

Hence every set of m Columns of the augmented matrix $Ab = [A, b]$ is L.I.

Sufficient Condition :-

Here we Consider that every set of m Columns of the augmented matrix $Ab = [A, b]$ is L.I. obviously every set of m - Column vectors of A are L.I. which implies that the basic solution of the system $Ax = b$ exist.

Now ~~we~~ to prove that the basic solution is non-degenerate. We consider $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any m-column vectors of A which is L.I. and let $x_1, x_2, \ldots, x_m$ be the corresponding basic solution.

$$\therefore x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = b$$

$$\therefore -1 \cdot b + x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = 0$$

Now if $x_1 = 0$, then we have

$$-1 \cdot b + x_2\alpha_2 + \cdots + x_m\alpha_m = 0$$

which implies that the vectors $b, \alpha_2, \ldots, \alpha_m$ are L.D. which is a contradiction since we have assumed that every set of m columns (vectors) of the augmented matrix $Ab = [A, b]$ is L.I.

Therefore $x_1 \neq 0$

Similarly we can prove that none of $x_2, x_3, \ldots, x_m$ are zero.

Hence the basic solution is non-degenerate.

In this way we can show that every basic solution is non-degenerate.