# Abstract Algebra

## Simple extension of a field

Let $S$ be a non-empty subset of a field $K$, then the intersection of all subfields of $K$ containg $S$ is a smallest subfield of $K$.

If $K$ be an extension of $F$ and $S \subseteq K$. the subfield of $K$ generated by $F \cup S$ is called a subfield of $K$ generated by $S$ over $F$ which is denoted by $F(S)$. If $S$ is finite and let $S = \{a_1, a_2, \ldots, a_n\}$. then $F(S) = F(a_1, a_2, \ldots, a_n)$

**Definition 1.** A field $K$ is said to be finitely generated over $F$ if there exists a finite number of elements say $a_1, a_2, \ldots, a_n \in K$ such that $K = F(a_1, a_2, \ldots, a_n)$.

In particular, if $a$ be an arbitrary element of $K$. then a subfield of $K$ generated by $F \cup \{a\}$ is a subfield of $K$ generated by $\{a\}$ over $F$ and it is denoted by $F(a)$. Thus $F(a)$ is the smallest subfield of $K$ containing $F$ and $a$, which is obtained by adjoining $a$ to $F$ and this process of adjoining is called an adjunction of $a$ to $F$.

**Definition 2** An extension $K$ of a field $F$ is called a simple extension of $F$ if $K = F(a)$, where $a \in K$. This element $a$ is known as a primitive element of $K$ over $F$.

For example $\mathbb{Q}(\sqrt{2})$ is a simple extension of the field $\mathbb{Q}$ of all rational numbers.

# Algebraic extension of a field

**Definition 1.** Let $F[x]$ be the set of all polynomials defined over $F$. Then an element $\alpha \in K$ (the extension of $F$) is said to be a root of a polynomial $f(x) \in F(x)$ if $f(\alpha) = 0$.

**Definition 2** (Algebraic element in a field extension). Let $K$ be an extension of field $F$. Then an element $\alpha \in K$ is said to be algebraic over $F$ if $\alpha$ is a root of a non-zero, non constant polynomial $f(x)$ of $F(x)$. If $\alpha$ is not algebraic, then it is transcendental.

**Definition 3** (Monic polynomial). A non-zero polynomial $f(x) \in F(x)$ is said to be monic polynomial over $F$. If the leading coefficient (the coefficient of highest power of $x$ in $f(x)$ is $1 \in F$.

**Definition 4** (Minimal polynomial) Let $K$ be an extension of a field $F$ and let $\alpha$ be an algebraic element of $K$. Then a polynomial of smallest degree $F$ satisfied by $\alpha$ is called a minimal polynomial of $\alpha$ over $F$.

If the degree of minimal polynomial of $\alpha$ over $F$ is $n$, then $\alpha$ is said to be an algebraic element of degree $n$.