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Theory of Equations

Theorem:- If $\frac{p}{q}$, where p and q are integers, prime to each other be a rational root of a polynomial equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-2} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are integers then p divides a_n and q divides a_0.

Proof:- since $\frac{p}{q}$ is a root of $f(x)=0$, then $f\left(\frac{p}{q}\right)=0$

This $\Rightarrow a_0\left[\frac{p}{q}\right]^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0$

$$\Rightarrow a_0 p^n + a_1 p^{n-1} q + \cdots + a_{n-1} p \cdot q^{n-1} + a_n q^n = 0 \quad \text{——— (1)}$$

$$\Rightarrow a_0 p^n = -q\{a_1 p^{n-1} + a_2 p^{n-2} q + \cdots + a_{n-1} p q^{n-2} + a_n q^{n-1}\}$$

$\Rightarrow q$ is a divisor of $a_0 p^n$.

But q is prime to p and hence it is prime to p^n. Therefore q is a divisor of a_0. Again the equation (1) can be written as

$$a_n q^n = -p\left\{a_0 p^{n-1} + a_1 p^{n-2} q + \cdots + a_{n-1} q^{n-1}\right\}$$

$\Rightarrow p$ is a divisor of $a_n q^n$.

But p and q are prime and hence p and q^n are prime to each other i.e., p cannot divide q^n.
Therefore p is a divisor of a_n.
Hence the theorem.

Cor:- If the leading coefficients of $f(x)$ in the above theorem be unity, then the rational roots of $f(x)=0$ are integers and divisors of a_n.

For, let $\frac{p}{q}$ be a rational root of $f(x)=0$.
Then by the above theorem,
q divides 1 and p divides a_n
Hence the rational roots are integers which divide a_n.

Q.) Find all the rational roots of the equation

$$2x^4 - 11x^3 + 17x^2 - 11x + 15 = 0$$

Sol:- Let p/q in its lowest term be the root of the given equation.
Then p must be a divisor of 15 and q a divisor of 2.
It follows that the values of p are

$$p = \pm 1, \pm 3, \pm 5, \pm 15$$

and those of q are $q = 1, 2$.

Thus the possible rational roots are

$\pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}, \pm 5, \pm \frac{5}{2}, \pm 15, \pm \frac{15}{2}$

It can be verified (by using synthetic division) that $x = 3$ and $x = 5/2$ are the only roots.

Hence $x = 3$ and $5/2$ are the only rational roots.

Thus, $f(x) = (x-3)(2x-5)(x^2+1)$