Theory of Equations

Division Alogrithm for polynomials

For any two polynomials $f(x) \neq 0$ and $g(x) \neq 0$, there exist uniquely two polynomials $q(x)$ and $r(x)$ such that $f(x) = g(x) \cdot q(x) + r(x)$ where either $r(x) = 0$ or $\deg . r(x) < \deg . g(x)$

According to this theorem if we divide $f(x)$ by $g(x)$, then the quotient $q(x)$ and the remainder $r(x)$ are unique.

Let $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$;

$a_0 \neq 0$

$g(x) = b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m$; $b_0 \neq 0$

(i) If $n < m$, then we can write
$f(x) = g(x) \cdot 0 + f(x)$ so that $q(x) = 0$ and $r(x) = f(x)$.

In this case we cannot divide $f(x)$ by $g(x)$.

(ii) If $n \geq m$, then we divide
$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$

by $g(x) = b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \dots + b_{m-1} x + b_m$;

Thus, $(b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m)(a_0/b_0) x^{n-m}$

$a_0 x^n + a_1 x^{n-1} + \dots + a_m x^{n-m} + \dots + a_{n-1} x + a_n$

$$a_0 x^n + \frac{a_0 b_1}{b_0} x^{n-1} + \cdots + \frac{a_0}{b_0} b_m x^{n-m}$$

$$\left(a_1 - \frac{a_0 b_1}{b_0}\right) x^{n-1} + \cdots + \left(a_m - \frac{a_0 b_m}{b_0}\right) x^{n-m} + a_{m-1} x^{n-m-1} + \cdots + a_{n-1} x + a_n$$

Therefore $f(x) = \frac{a_0}{b_0} x^{n-m} \cdot g(x) + r_1(x)$

where $r_1(x) = \left[a_1 - \frac{a_0 b_1}{b_0}\right] x^{n-1} + \cdots + a_{n-1} x + a_n$

so that $\deg. r_1(x) = n-1 < n$

i.e. $f(x) = q_1(x) g(x) + r_1(x)$ —— (1)

where $q_1(x) = (a_0/b_0) x^{n-m}$ and $\deg. r_1(x) = n-1 < n$

Let us assume that the theorem holds for degree less than n. We shall show by induction that the theorem holds for degree n.

Applying the result of the theorem to the polynomial $r_1(x)$ whose degree is $< n$, we can write $r_1(x) = g(x) \cdot h_1(x) + r(x)$ —— (2)

where $\deg. r(x) < \deg. g(x)$

Substituting this value of $r_1(x)$ in (1), we get

$$f(x) = q_1(x) g(x) + g(x) h_1(x) + r(x)$$

$$= \{q_1(x) + h_1(x)\} g(x) + r(x)$$

$$= q(x) g(x) + r(x).$$

Hence the division algorithm holds for the polynomial $f(x)$ of degree n. Thus it is true for polynomial of any degree.

Now we show that $q(x)$ and $r(x)$ are unique.

Let us assume that there exists a second pair $q'(x)$ and $r'(x)$ such that $f(x) = q'(x) \cdot g(x) + r'(x)$ where $r'(x)$ is either zero or $\deg . r'(x) < \deg . g(x)$

Then $q(x) \cdot g(x) + r(x) = q'(x) \cdot g(x) + r'(x)$

$$\Rightarrow g(x)\{q(x) - q'(x)\}$$
$$= r'(x) - r(x)$$

If $q(x) - q'(x) \neq 0$, then the degree of L.H.S. is $\geq \deg . g(x) \geq m$. But the R.H.S. is either zero or of $\deg . < \deg . g(x) < m$.

Therefore it is necessary that $q(x) - q'(x) = 0$

i.e. $q(x) = q'(x)$

Therefore $r'(x) = r(x)$.

Hence $q(x)$ and $r(x)$ are unique.

<u>Cor. 1</u> <u>Remainder theorem</u>:- If we put $g(x) = (x - a)$ (a polynomial of first degree) in the above division algorithm